CHAPTER 7

CHAPTER 7 Similarity

Chapter Outline

In this chapter, we will start with a review of ratios and proportions. Second, we will introduce the concept of similarity. Two figures are similar if they have the same shape, but not the same size. We will apply similarity to polygons, quadrilaterals and triangles. Then, we will extend this concept to proportionality with parallel lines and dilations. Finally, there is an extension about self-similarity, or fractals, at the end of the chapter.

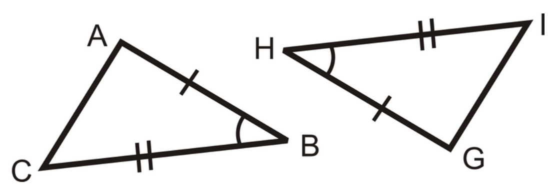
7.1 Ratios and Proportions

Learning Objectives

- Write, simplify, and solve ratios and proportions.
- Use ratios and proportions in problem solving.

Review Queue

a. Are the two triangles congruent? If so, how do you know?



b. If $AC = 5$, what is GI? What is the reason?
c. How many inches are in a foot? In a yard? In 3 yards?
d. How many cups are in a pint? In a quart? In a gallon? In 7 quarts?

Know What? You want to make a scale drawing of your room and furniture for a little redecorating. Your room measures 12 feet by 12 feet. Also in your room is a twin bed (36 in by 75 in), a desk (4 feet by 2 feet), and a chair (3 feet by 3 feet). You decide to scale down your room to 8 in by 8 in, so the drawing fits on a piece of paper. What size should the bed, desk and chair be? Draw an appropriate layout for the furniture within the room. *Do not round your answers.*

Using Ratios

Ratio: A way to compare two numbers. Ratios can be written: $\frac{a}{b}, a : b$, and a to b.

Example 1: The total bagel sales at a bagel shop for Monday is in the table below. What is the ratio of cinnamon raisin bagels to plain bagels?

TABLE 7.1:

Type of Bagel	*Number Sold*
Plain	80
Cinnamon Raisin	30
Sesame	25
Jalapeno Cheddar	20
Everything	45
Honey Wheat	50

Solution: The ratio is $\frac{30}{80}$, 30:80, or 30 to 80. Depending on the problem, ratios are usually written in simplest form, which means to reduce the ratio. The answer is then $\frac{3}{8}$, 3:8, or 3 to 8.

Example 2: What is the ratio, in simplest form, of Honey Wheat bagels to total bagels sold?

Solution: Remember that order matters. Because the Honey Wheat is listed first, that is the number that comes first in the ratio (on in the numerator of the fraction). Find the total number of bagels sold.

$$80+30+25+20+45+50=250$$

The ratio is then $\frac{50}{250}=\frac{1}{5}$, 1:5, or 1 to 5.

We call the ratio 50:250 and 1:5 ***equivalent*** because one reduces to the other.

In some problems you may need to write a ratio of more than two numbers. For example, the ratioof the number of cinnamon raisin bagels to sesame bagels to jalapeno cheddar bagels is 30:25:20 or 6:5:4.

Measurements are used a lot with ratios and proportions. For example, how many feet are in 2 miles? How many inches are in 4 feet? You will need to know these basic measurements.

Example 3: Simplify the following ratios.

a) $\frac{7\ ft}{14\ in}$

b) $9m : 900cm$

c) $\frac{4\ gal}{16\ gal}$

Solution: Change these so that they are in the same units.

a) $\frac{7\ \cancel{ft}}{14\ \cancel{in}}\cdot\frac{12\ \cancel{in}}{1\ \cancel{ft}}=\frac{84}{14}=\frac{6}{1}$

Notice that the inches cancel each other out. ***All ratios should not have units once simplified.***

b) It is easier to simplify ratios when they are written as a fraction. $\frac{9\ m}{900\ cm}\cdot\frac{100\ cm}{1\ m}=\frac{900}{900}=\frac{1}{1}$

c) $\frac{4\ gal}{16\ gal}=\frac{1}{4}$

Example 4: A talent show features dancers and singers. The ratio of dancers to singers is 3:2. There are 30 performers total, how many singers are there?

Solution: 3:2 is a reduced ratio, so there is a whole number, n, that we can multiply both by to find the total number in each group.

$$\begin{aligned}\text{dancers}=3n, \text{singers}=2n \ \longrightarrow\ 3n+2n&=30\\ 5n&=30\\ n&=6\end{aligned}$$

Therefore, there are $3\cdot6=18$ dancers and $2\cdot6=12$ singers. To double-check, $18+12=30$ total performers.

Proportions

Proportion: When two ratios are set equal to each other.

Example 4: Solve the proportions.

a) $\frac{4}{5} = \frac{x}{30}$

b) $\frac{y+1}{8} = \frac{5}{20}$

c) $\frac{6}{5} = \frac{2x+5}{x-2}$

Solution: To solve a proportion, you need to ***cross-multiply***.

a)

$$\begin{aligned} \frac{4}{5} &= \frac{x}{30} \\ 4 \cdot 30 &= 5 \cdot x \\ 120 &= 5x \\ 24 &= x \end{aligned}$$

b)

$$\begin{aligned} \frac{y+1}{8} &= \frac{5}{20} \\ (y+1) \cdot 20 &= 5 \cdot 8 \\ 20y + 20 &= 40 \\ 20y &= 20 \\ y &= 1 \end{aligned}$$

c)

$$\begin{aligned} \frac{6}{5} &= \frac{2x+4}{x-2} \\ 6 \cdot (x-2) &= 5 \cdot (2x+4) \\ 6x-12 &= 10x+20 \\ -32 &= 4x \\ -8 &= x \end{aligned}$$

In proportions, the blue numbers are called the ***means*** and the orange numbers are called the ***extremes***. For the proportion to be true, the product of the means must equal the product of the extremes. This can be generalized in the Cross-Multiplication Theorem.

Cross-Multiplication Theorem: Let a, b, c, and d be real numbers, with $b \neq 0$ and $d \neq 0$. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

The proof of the Cross-Multiplication Theorem is an algebraic proof. Recall that multiplying by $\frac{2}{2}$, $\frac{b}{b}$, or $\frac{d}{d} = 1$ because it is the same number divided by itself $(b \div b = 1)$.

Proof of the Cross-Multiplication Theorem

$$\frac{a}{b} = \frac{c}{d} \qquad \text{Multiply the left side by } \frac{d}{d} \text{ and the right side by } \frac{b}{b}.$$

$$\frac{a}{b} \cdot \frac{d}{d} = \frac{c}{d} \cdot \frac{b}{b}$$

$$\frac{ad}{bd} = \frac{bc}{bd} \qquad \text{The denominators are the same, so the numerators are equal.}$$

$$ad = bc$$

Think of the Cross-Multiplication Theorem as a shortcut. Without this theorem, you would have to go through all of these steps every time to solve a proportion.

Example 5: Your parents have an architect's drawing of their home. On the paper, the house's dimensions are 36 in by 30 in. If the shorter length of your parents' house is actually 50 feet, what is the longer length?

Solution: Set up a proportion. If the shorter length is 50 feet, then it will line up with 30 in. It does not matter which numbers you put in the numerators of the fractions, as long as they line up correctly.

$$\frac{30}{36} = \frac{50}{x} \longrightarrow 1800 = 30x$$
$$60 = x$$

So, the dimension of your parents' house is 50 ft by 60 ft.

Properties of Proportions

The Cross-Multiplication Theorem has several sub-theorems that follow from its proof. The formal term is ***corollary.***

Corollary: A theorem that follows quickly, easily, and directly from another theorem.

Below are three corollaries that are immediate results of the Cross Multiplication Theorem and the fundamental laws of algebra.

Corollary 7-1: If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

Corollary 7-2: If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{d}{b} = \frac{c}{a}$.

Corollary 7-3: If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

In other words, a true proportion is also true if you switch the means, switch the extremes, or flip it upside down. Notice that you will still end up with $ad = bc$ after cross-multiplying for all three of these corollaries.

Example 6: Suppose we have the proportion $\frac{2}{5} = \frac{14}{35}$. Write down the other three true proportions that follow from this one.

Solution: First of all, we know this is a true proportion because you would multiply $\frac{2}{5}$ by $\frac{7}{7}$ to get $\frac{14}{35}$. Using the three corollaries, we would get:

a. $\frac{2}{14} = \frac{5}{35}$
b. $\frac{35}{5} = \frac{14}{2}$
c. $\frac{5}{2} = \frac{35}{14}$

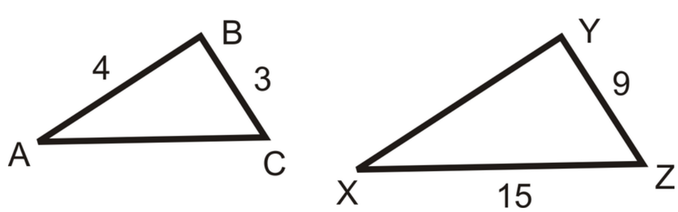
If you cross-multiply all four of these proportions, you would get $70 = 70$ for each one.

Corollary 7-4: If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

Corollary 7-5: If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$.

Example 7: In the picture, $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$.

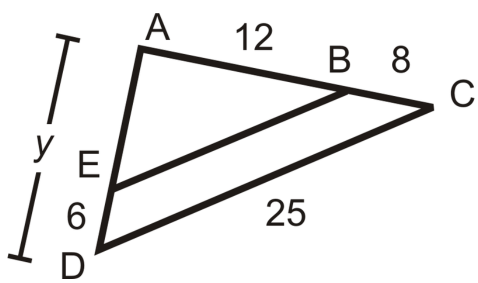
Find the measures of AC and XY.



Solution: This is an example of an *extended* proportion. Substituting in the numbers for the sides we know, we have $\frac{4}{XY} = \frac{3}{9} = \frac{AC}{15}$. Separate this into two different proportions and solve for XY and AC.

$$\begin{aligned} \frac{4}{XY} &= \frac{3}{9} \\ 36 &= 3(XY) \\ XY &= 12 \end{aligned} \qquad\qquad \begin{aligned} \frac{3}{9} &= \frac{AC}{15} \\ 9(AC) &= 45 \\ AC &= 5 \end{aligned}$$

Example 8: In the picture, $\frac{ED}{AD} = \frac{BC}{AC}$. Find y.



Solution: Substituting in the numbers for the sides we know, we have

$$\begin{aligned} \frac{6}{y} = \frac{8}{12+8}. \longrightarrow 8y &= 6(20) \\ y &= 15 \end{aligned}$$

Example 9: If $\frac{AB}{BE} = \frac{AC}{CD}$ in the picture above, find BE.

Solution:

$$\begin{aligned} \frac{12}{BE} = \frac{20}{25} \longrightarrow 20(BE) &= 12(25) \\ BE &= 15 \end{aligned}$$

Know What? Revisited Everything needs to be scaled down by a factor of $\frac{1}{18}$ $(144\ in \div 8\ in)$. Change everything into inches and then multiply by the scale factor.

Bed: 36 in by 75 in $\longrightarrow$ 2 in by 4.167 in

Desk: 48 in by 24 in $\longrightarrow$ 2.67 in by 1.33 in

Chair: 36 in by 36 in $\longrightarrow$ 2 in by 2 in

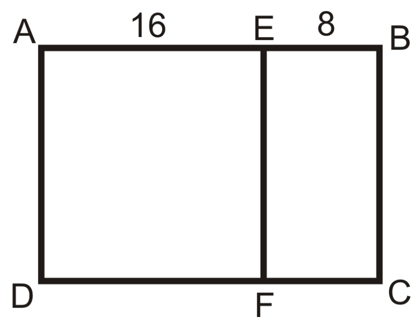
There are several layout options for these three pieces of furniture. Draw an 8 in by 8 in square and then the appropriate rectangles for the furniture. Then, cut out the rectangles and place inside the square.

Review Questions

1. The votes for president in a club election were: Smith : 24 Munoz : 32 Park : 20 Find the following ratios and write in simplest form.

a. Votes for Munoz to Smith
b. Votes for Park to Munoz
c. Votes for Smith to total votes
d. Votes for Smith to Munoz to Park

Use the picture to write the following ratios for questions 2-6.



$AEFD$ is a square

$ABCD$ is a rectangle

2. $AE : EF$
3. $EB : AB$
4. $DF : FC$
5. $EF : BC$
6. Perimeter $ABCD$: Perimeter $AEFD$: Perimeter $EBCF$
7. The measures of the angles of a triangle are have the ratio 3:3:4. What are the measures of the angles?
8. The lengths of the sides in a triangle are in a 3:4:5 ratio. The perimeter of the triangle is 36. What are the lengths of the sides?
9. The length and width of a rectangle are in a 3:5 ratio. The perimeter of the rectangle is 64. What are the length and width?
10. The length and width of a rectangle are in a 4:7 ratio. The perimeter of the rectangle is 352. What are the length and width?
11. The ratio of the short side to the long side in a parallelogram is 5:9. The perimeter of the parallelogram is 112. What are the lengths of the sides?
12. The length and width of a rectangle are in a 3:11 ratio. The area of the rectangle is 528. What are the length and width of the rectangle?
13. ***Writing*** Explain why $\frac{a+b}{b} = \frac{c+d}{d}$ is a valid proportion. HINT: Cross-multiply and see if it equals $ad = bc$.
14. ***Writing*** Explain why $\frac{a-b}{b} = \frac{c-d}{d}$ is a valid proportion. HINT: Cross-multiply and see if it equals $ad = bc$.

Solve each proportion.

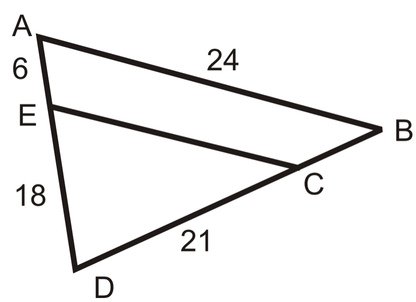
15. $\frac{x}{10} = \frac{42}{35}$
16. $\frac{x}{x-2} = \frac{5}{7}$
17. $\frac{6}{9} = \frac{y}{24}$
18. $\frac{x}{9} = \frac{16}{x}$
19. $\frac{y-3}{8} = \frac{y+6}{5}$
20. $\frac{20}{z+5} = \frac{16}{7}$
21. Shawna drove 245 miles and used 8.2 gallons of gas. At the same rate, if she drove 416 miles, how many gallons of gas will she need? Round to the nearest tenth.

22. The president, vice-president, and financial officer of a company divide the profits is a 4:3:2 ratio. If the company made $1,800,000 last year, how much did each person receive?
23. Many recipes describe ratios between ingredients. For example, one recipe for paper mache paste suggests 3 parts flour to 5 parts water. If we have one cup of flour, how much water should we add to make the paste?
24. A recipe for krispy rice treats calls for 6 cups of rice cereal and 40 large marshmallows. You want to make a larger batch of goodies and have 9 cups of rice cereal. How many large marshmallows do you need? However, you only have the miniature marshmallows at your house. You find a list of substitution quantities on the internet that suggests 10 large marshmallows are equivalent to 1 cup miniatures. How many cups of miniatures do you need?

Given the true proportion, $\frac{10}{6} = \frac{15}{d} = \frac{x}{y}$ and d, x, and y are nonzero, determine if the following proportions are also true.

25. $\frac{10}{y} = \frac{x}{6}$
26. $\frac{15}{10} = \frac{d}{6}$
27. $\frac{6+10}{10} = \frac{y+x}{x}$
28. $\frac{15}{x} = \frac{y}{d}$

For questions 24-27, $\frac{AE}{ED} = \frac{BC}{CD}$ and $\frac{ED}{AD} = \frac{CD}{DB} = \frac{EC}{AB}$.



29. Find DB.
30. Find EC.
31. Find CB.
32. Find AD.

Review Queue Answers

a. Yes, they are congruent by SAS.
b. $GI = 5$ by CPCTC
c. 12 in = 1 ft, 36 in = 3 ft, 108 in = 3 yrds
d. 2c = 1 pt, 4c = 1 qt, 16 c = 4 qt = 1 gal, 28c = 7 qt

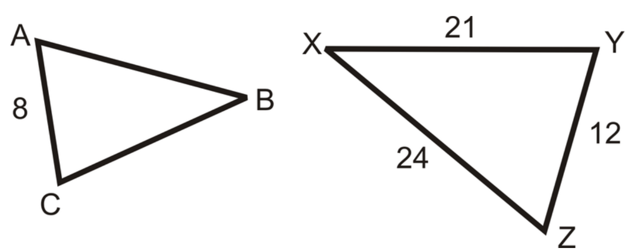
7.2 Similar Polygons

Learning Objectives

- Recognize similar polygons.
- Identify corresponding angles and sides of similar polygons from a similarity statement.
- Calculate and apply scale factors.

Review Queue

a. Solve the proportions.
   a. $\frac{6}{x} = \frac{10}{15}$
   b. $\frac{4}{7} = \frac{2x+1}{42}$
   c. $\frac{5}{8} = \frac{x-2}{2x}$
b. In the picture, $\frac{AB}{XZ} = \frac{BC}{XY} = \frac{AC}{YZ}$.
   a. Find AB.
   b. Find BC.

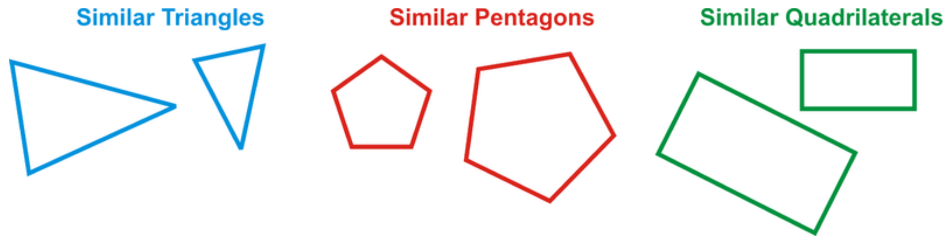


Know What? A baseball diamond is a square with 90 foot sides. A softball diamond is a square with 60 foot sides. Are the two diamonds similar? If so, what is the scale factor? Explain your answer.

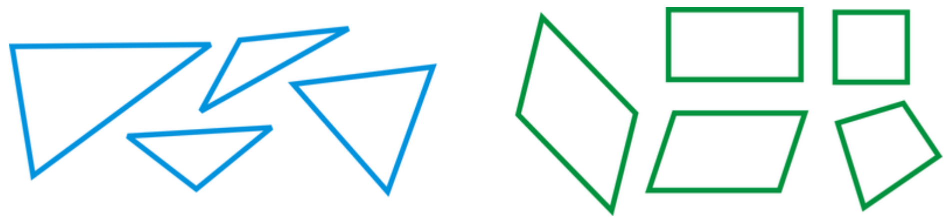
Similar Polygons

Similar Polygons: Two polygons with the same shape, but not the same size.

Think about similar polygons as an enlargement or shrinking of the same shape. So, more specifically, similar polygons have to have the same number of sides, the corresponding angles are congruent, and the corresponding sides are proportional. The symbol $\sim$ is used to represent similar. Here are some examples:



These polygons are not similar:

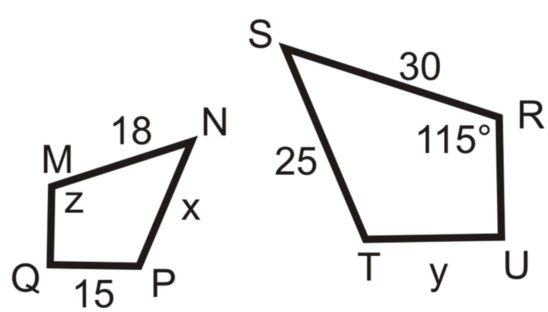


Example 1: Suppose $\triangle ABC \sim \triangle JKL$. Based on the similarity statement, which angles are congruent and which sides are proportional?

Solution: Just like a congruence statement, the congruent angles line up within the statement. So, $\angle A \cong \angle J, \angle B \cong \angle K$, and $\angle C \cong \angle L$. The same is true of the proportional sides. We write the sides in a proportion, $\frac{AB}{JK} = \frac{BC}{KL} = \frac{AC}{JL}$.

Because of the corollaries we learned in the last section, the proportions in Example 1 could be written several different ways. For example, $\frac{AB}{BC} = \frac{JK}{KL}$. Make sure to line up the corresponding proportional sides.

Example 2: $MNPQ \sim RSTU$. What are the values of x, y and z?



Solution: In the similarity statement, $\angle M \cong \angle R$, so $z = 115^\circ$. For x and y, set up a proportion.

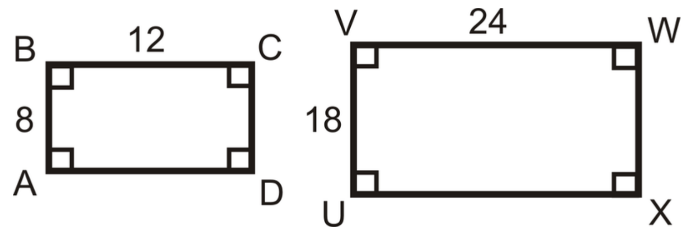
$$\frac{18}{30} = \frac{x}{25} \qquad \frac{18}{30} = \frac{15}{y}$$

$$450 = 30x \qquad 450 = 18y$$

$$x = 15 \qquad y = 25$$

Specific types of triangles, quadrilaterals, and polygons will always be similar. For example, because all the angles and sides are congruent, ***all equilateral triangles are similar.*** For the same reason, ***all squares are similar.*** We can take this one step further and say that all regular polygons (with the same number of sides) are similar.

Example 3: $ABCD$ is a rectangle with length 12 and width 8. $UVWX$ is a rectangle with length 24 and width 18. Are these two rectangles similar?



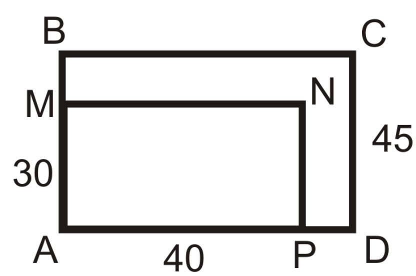
Solution: Draw a picture. First, all the corresponding angles need to be congruent. In rectangles, all the angles are congruent, so this condition is satisfied. Now, let's see if the sides are proportional. $\frac{8}{12} = \frac{2}{3}, \frac{18}{24} = \frac{3}{4}$. $\frac{2}{3} \neq \frac{3}{4}$. This tells us that the sides are not in the same proportion, so the rectangles are not similar. We could have also set up the proportion as $\frac{12}{24} = \frac{1}{2}$ and $\frac{8}{18} = \frac{4}{9}$. $\frac{1}{2} \neq \frac{4}{9}$, so you would end up with the same conclusion.

Scale Factors

If two polygons are similar, we know the lengths of corresponding sides are proportional. If k is the length of a side in one polygon, and m is the length of the corresponding side in the other polygon, then the ratio $\frac{k}{m}$ is the ***scale factor*** relating the first polygon to the second.

Scale Factor: In similar polygons, the ratio of one side of a polygon to the corresponding side of the other.

Example 5: $ABCD \sim AMNP$. Find the scale factor and the length of BC.



Solution: Line up the corresponding proportional sides. $AB : AM$, so the scale factor is $\frac{30}{45} = \frac{2}{3}$ or $\frac{3}{2}$. Because BC is in the bigger rectangle, we will multiply 40 by $\frac{3}{2}$ because it is greater than 1. $BC = \frac{3}{2}(40) = 60$.

Example 6: Find the perimeters of $ABCD$ and $AMNP$. Then find the ratio of the perimeters.

Solution: Perimeter of $ABCD = 60 + 45 + 60 + 45 = 210$

Perimeter of $AMNP = 40 + 30 + 40 + 30 = 140$

The ratio of the perimeters is 140:210, which reduces to 2:3.

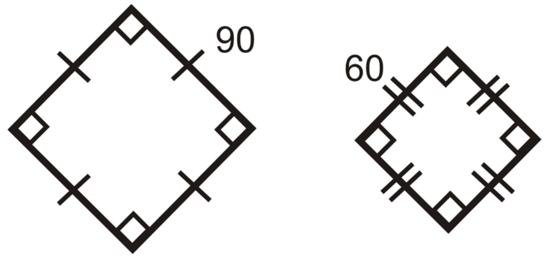
Theorem 7-2: The ratio of the perimeters of two similar polygons is the same as the ratio of the sides.

In addition the perimeter being in the same ratio as the sides, all parts of a polygon are in the same ratio as the sides. This includes diagonals, medians, midsegments, altitudes, and others.

Example 7: $\triangle ABC \sim \triangle MNP$. The perimeter of $\triangle ABC$ is 150 and $AB = 32$ and $MN = 48$. Find the perimeter of $\triangle MNP$.

Solution: From the similarity statement, AB and MN are corresponding sides. So, the scale factor is $\frac{32}{48} = \frac{2}{3}$ or $\frac{3}{2}$. The perimeter of $\triangle MNP$ is $\frac{2}{3}(150) = 100$.

Know What? Revisited All of the sides in the baseball diamond are 90 feet long and 60 feet long in the softball diamond. This means all the sides are in a $\frac{90}{60} = \frac{3}{2}$ ratio. All the angles in a square are congruent, all the angles in both diamonds are congruent. The two squares are similar and the scale factor is $\frac{3}{2}$.

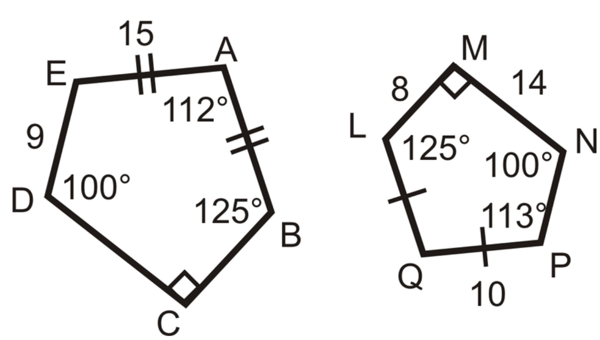


Review Questions

Determine if the following statements are true or false.

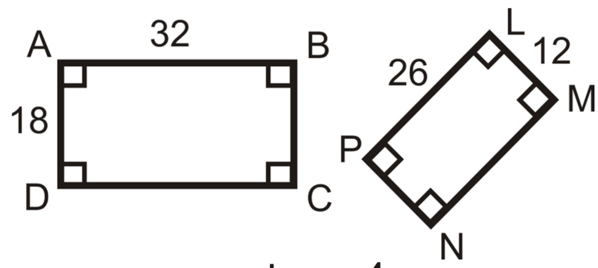
1. All equilateral triangles are similar.
2. All isosceles triangles are similar.
3. All rectangles are similar.
4. All rhombuses are similar.
5. All squares are similar.
6. All congruent polygons are similar.
7. All similar polygons are congruent.
8. All regular pentagons are similar.
9. $\triangle BIG \sim \triangle HAT$. List the congruent angles and proportions for the sides.
10. If $BI = 9$ and $HA = 15$, find the scale factor.
11. If $BG = 21$, find HT.
12. If $AT = 45$, find IG.
13. Find the perimeter of $\triangle BIG$ and $\triangle HAT$. What is the ratio of the perimeters?

Use the picture to the right to answer questions 14-18.



14. Find $m\angle E$ and $m\angle Q$.
15. $ABCDE \sim QLMNP$, find the scale factor.
16. Find BC.
17. Find CD.
18. Find NP.

Determine if the following triangles and quadrilaterals are similar. If they are, write the similarity statement.



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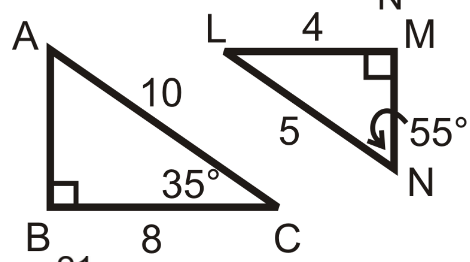

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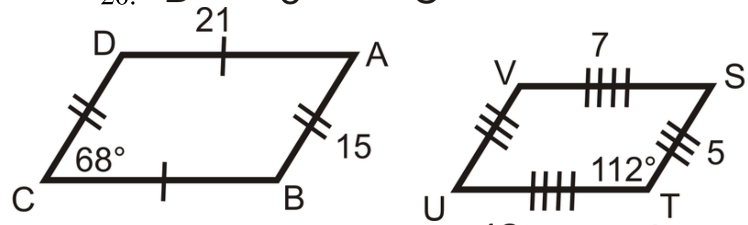

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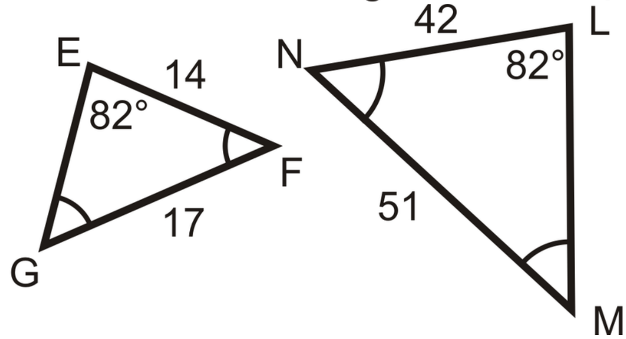
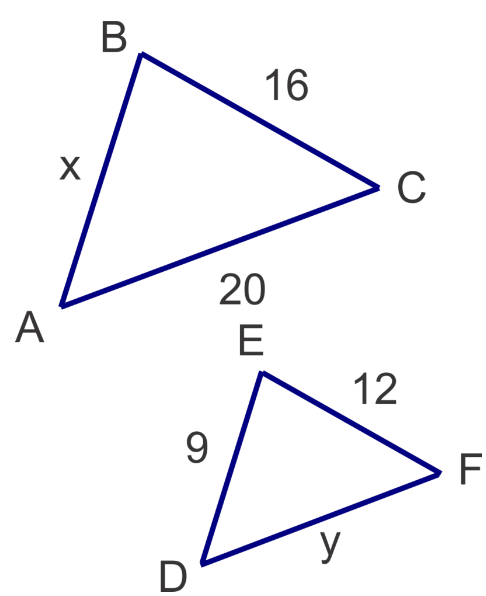
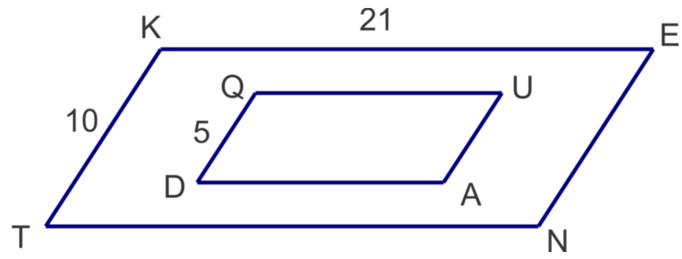


22.

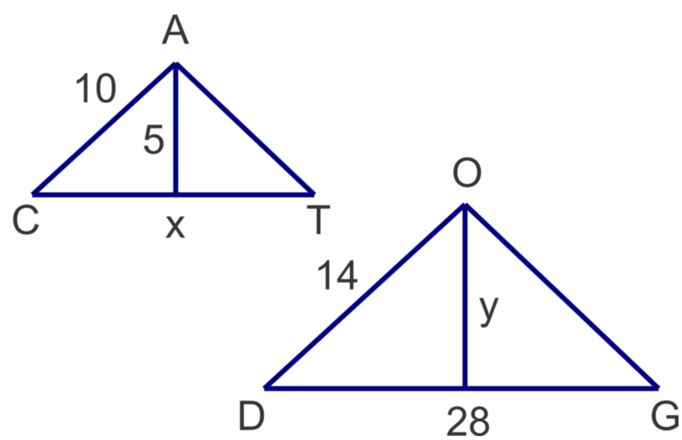
23. $\triangle ABC \sim \triangle DEF$ Solve for x and y.



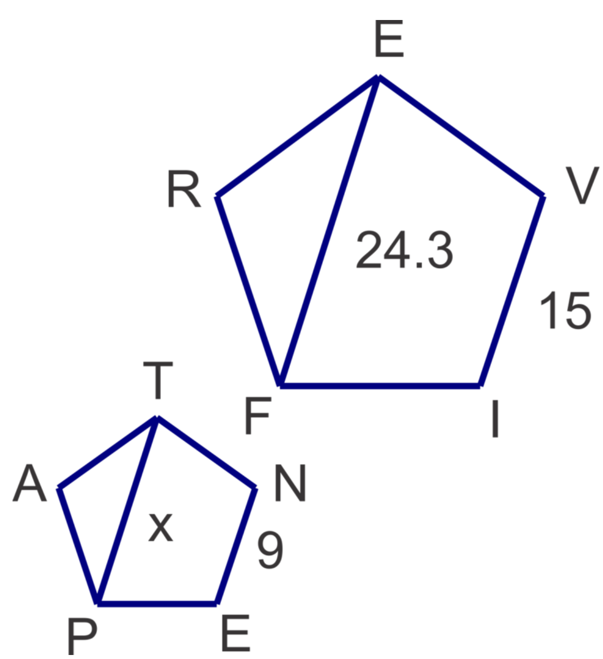
24. $QUAD \sim KENT$ Find the perimeter of $QUAD$.



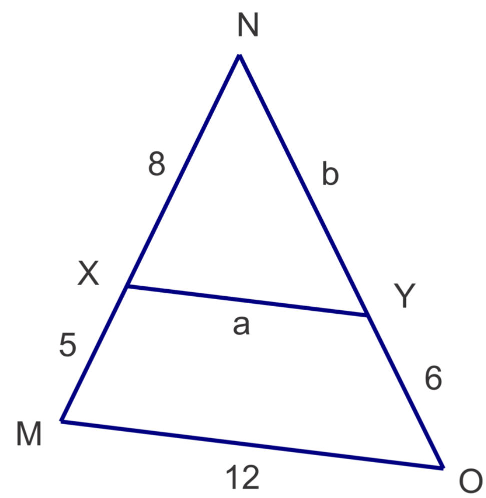
25. $\triangle CAT \sim \triangle DOG$ Solve for x and y.



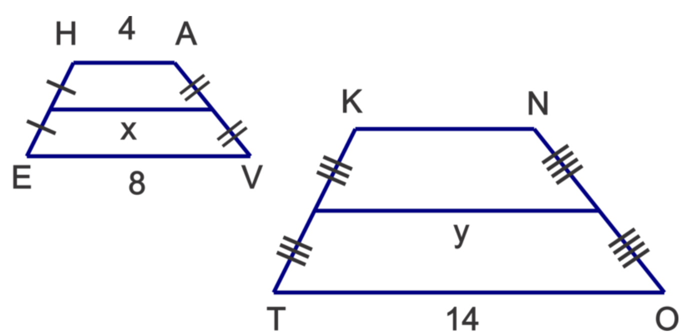
26. $PENTA \sim FIVER$ Solve for x.



27. $\triangle MNO \sim \triangle XNY$ Solve for a and b.



28. Trapezoids $HAVE \sim KNOT$ Solve for x and y.



29. Two similar octagons have a scale factor of $\frac{9}{11}$. If the perimeter of the smaller octagon is 99 meters, what is the perimeter of the larger octagon?
30. Two right triangles are similar. The legs of one of the triangles are 5 and 12. The second right triangle has a hypotenuse of length 39. What is the scale factor between the two triangles?
31. What is the area of the smaller triangle in problem 30? What is the area of the larger triangle in problem 30? What is the ratio of the areas? How does it compare to the ratio of the lengths (or scale factor)? Recall that the area of a triangle is $A = \frac{1}{2}bh$.

Review Queue Answers

a. $x = 9$
b. $x = 11.5$
c. $x = 8$

a. $AB = 16$
b. $BC = 14$

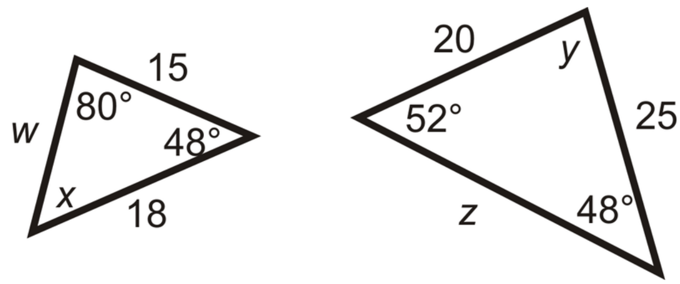
7.3 Similarity by AA

Learning Objectives

- Determine whether triangles are similar.
- Understand AA for similar triangles.
- Solve problems involving similar triangles.

Review Queue

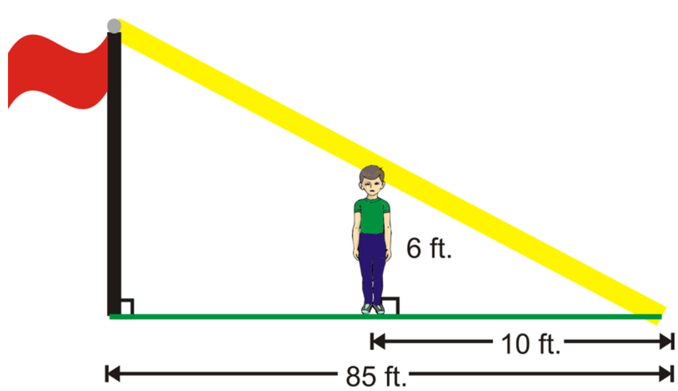
a. a. Find the measures of x and y.
   b. The two triangles are similar. Find w and z.



b. Use the true proportion $\frac{6}{8} = \frac{x}{28} = \frac{27}{y}$ to answer the following questions.

   a. Find x and y.
   b. Write another true proportion.
   c. Is $\frac{28}{8} = \frac{6+x}{12}$ true? If you solve for x, is it the same as in part a?

Know What? George wants to measure the height of a flagpole. He is 6 feet tall and his shadow is 10 feet long. At the same time, the shadow of the flagpole was 85 feet long. How tall is the flagpole?



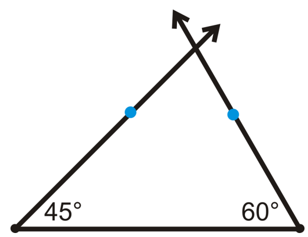
Angles in Similar Triangles

The Third Angle Theorem states if two angles are congruent to two angles in another triangle, the third angles are congruent too. Because a triangle has 180°, the third angle in any triangle is 180° minus the other two angle measures. Let's investigate what happens when two different triangles have the same angle measures. We will use Investigation 4-4 (Constructing a Triangle using ASA) to help us with this.

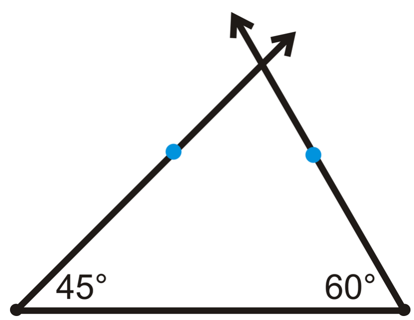
Investigation 7-1: Constructing Similar Triangles

Tools Needed: pencil, paper, protractor, ruler

a. Draw a 45° angle. Extend the horizontal side and then draw a 60° angle on the other side of this side. Extend the other side of the 45° angle and the 60° angle so that they intersect to form a triangle. What is the measure of the third angle? Measure the length of each side.



b. Repeat Step 1 and make the horizontal side between the 45° and 60° angle at least 1 inch longer than in Step 1. This will make the entire triangle larger. Find the measure of the third angle and measure the length of each side.

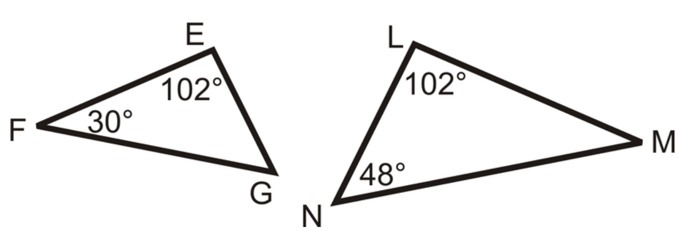


c. Find the ratio of the sides. Put the sides opposite the 45° angles over each other, the sides opposite the 60° angles over each other, and the sides opposite the third angles over each other. What happens?

AA Similarity Postulate: If two angles in one triangle are congruent to two angles in another triangle, the two triangles are similar.

The AA Similarity Postulate is a shortcut for showing that two ***triangles*** are similar. If you know that two angles in one triangle are congruent to two angles in another, which is now enough information to show that the two triangles are similar. Then, you can use the similarity to find the lengths of the sides.

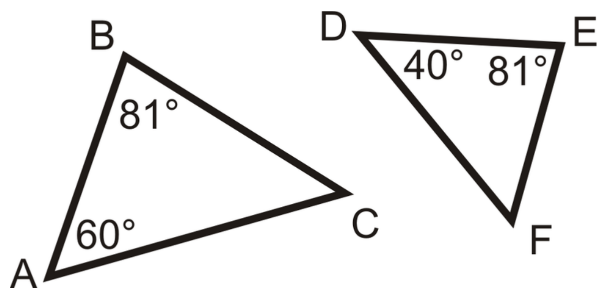
Example 1: Determine if the following two triangles are similar. If so, write the similarity statement.



Solution: Find the measure of the third angle in each triangle. $m\angle G = 48^\circ$ and $m\angle M = 30^\circ$ by the Triangle Sum Theorem. Therefore, all three angles are congruent, so the two triangles are similar. $\triangle FEG \sim \triangle MLN$.

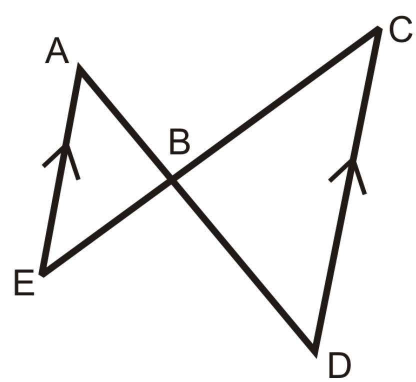
Example 2: Determine if the following two triangles are similar. If so, write the similarity statement.

Solution: $m\angle C = 39^\circ$ and $m\angle F = 59^\circ$. The angles are not equal, $\triangle ABC$ and $\triangle DEF$ are not similar.



Example 3: Are the following triangles similar? If so, write the similarity statement.

Solution: Because $\overline{AE} \parallel \overline{CD}$, $\angle A \cong \angle D$ and $\angle C \cong \angle E$ by the Alternate Interior Angles Theorem. Therefore, by the AA Similarity Postulate, $\triangle ABE \sim \triangle DBC$.

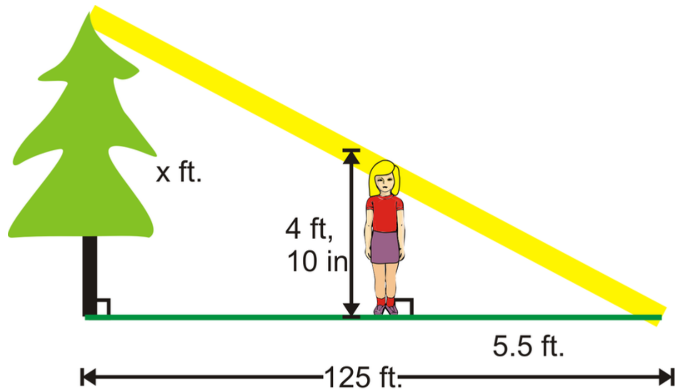


Indirect Measurement

An application of similar triangles is to measure lengths *indirectly.* The length to be measured would be some feature that was not easily accessible to a person, such as: the width of a river or canyon and the height of a tall object. To measure something indirectly, you would need to set up a pair of similar triangles.

Example 4: A tree outside Ellie's building casts a 125 foot shadow. At the same time of day, Ellie casts a 5.5 foot shadow. If Ellie is 4 feet 10 inches tall, how tall is the tree?

Solution: Draw a picture. From the picture to the right, we see that the tree and Ellie are parallel, therefore the two triangles are similar to each other. Write a proportion.



$$\frac{4ft, 10in}{xft} = \frac{5.5ft}{125ft}$$

Notice that our measurements are not all in the same units. Change both numerators to inches and then we can cross multiply.

$$\frac{58in}{xft} = \frac{66in}{125ft} \longrightarrow 58(125) = 66(x)$$
$$7250 = 66x$$
$$x \approx 109.85\ ft$$

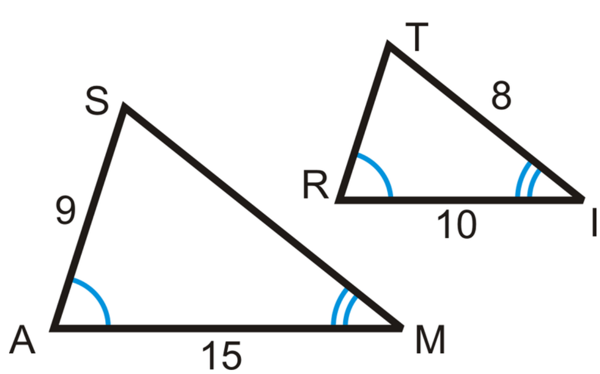
Know What? Revisited It is safe to assume that George and the flagpole stand vertically, making right angles with the ground. Also, the angle where the sun's rays hit the ground is the same for both. The two trianglesare similar. Set up a proportion.

$$\frac{10}{85} = \frac{6}{x} \longrightarrow 10x = 510$$
$$x = 51\ ft.$$

The height of the flagpole is 51 feet.

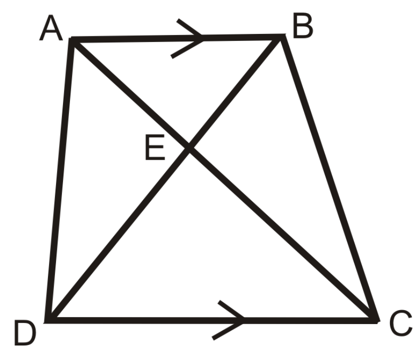
Review Questions

Use the diagram to complete each statement.



1. $\triangle SAM \sim \triangle$____
2. $\frac{SA}{?} = \frac{SM}{?} = \frac{?}{RI}$
3. $SM =$ ____
4. $TR =$ ____
5. $\frac{9}{?} = \frac{?}{8}$

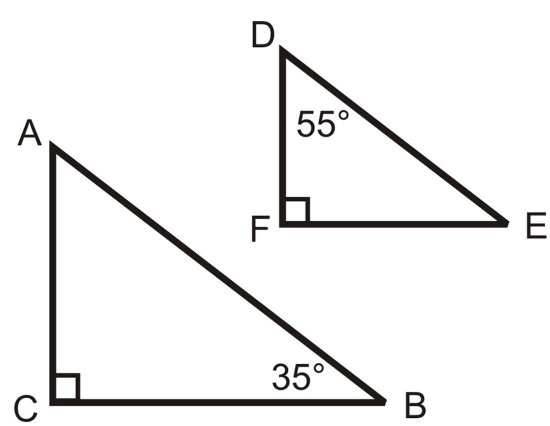
Answer questions 6-9 about trapezoid $ABCD$.



6. Name two similar triangles. How do you know they are similar?
7. Write a true proportion.
8. Name two other triangles that might *not* be similar.
9. If $AB = 10, AE = 7$, and $DC = 22$, find AC. Be careful!
10. ***Writing*** How many angles need to be congruent to show that two triangles are similar? Why?
11. ***Writing*** How do congruent triangles and similar triangles differ? How are they the same?

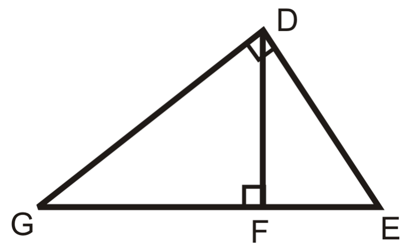
Use the triangles to the left for questions 5-9.

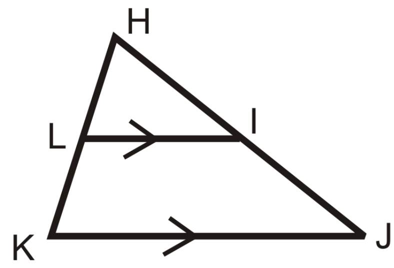
$AB = 20, DE = 15$, and $BC = k$.

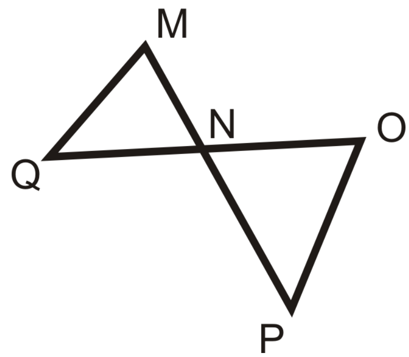


12. Are the two triangles similar? How do you know?
13. Write an expression for FE in terms of k.
14. If $FE = 12$, what is k?
15. Fill in the blanks: If an acute angle of a ________ triangle is congruent to an acute angle in another ________-triangle, then the two triangles are ________.

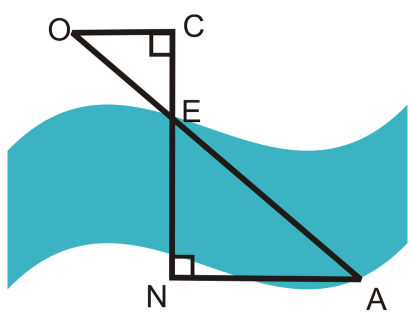
Are the following triangles similar? If so, write a similarity statement.

16. 

17. 

18. 

In order to estimate the width of a river, the following technique can be used. Use the diagram on the left.

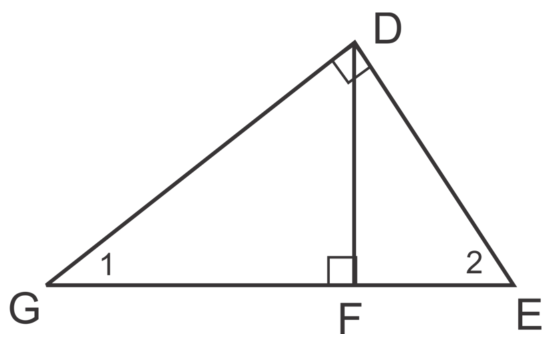


Place three markers, O, C, and E on the upper bank of the river. E is on the edge of the river and $\overline{OC} \perp \overline{CE}$. Go across the river and place a marker, N so that it is collinear with C and E. Then, walk along the lower bank of the river and place marker A, so that $\overline{CN} \perp \overline{NA}$. $OC = 50 \ feet, CE = 30 \ feet, NA = 80 \ feet$.

19. Is $\overline{OC} \ || \ \overline{NA}$? How do you know?
20. Is $\triangle OCE \sim \triangle ANE$? How do you know?
21. What is the width of the river? Find EN.
22. Can we find EA? If so, find it. If not, explain.
23. Janet wants to measure the height of her apartment building. She places a pocket mirror on the ground 20 ft from the building and steps backwards until she can see the top of the build in the mirror. She is 18 in from the mirror and her eyes are 5 ft 3 in above the ground. The angle formed by her line of sight and the ground is congruent to the angle formed by the reflection of the building and the ground. You may wish to draw a diagram to illustrate this problem. How tall is the building?

24. Sebastian is curious to know how tall the announcer's box is on his school's football field. On a sunny day he measures the shadow of the box to be 45 ft and his own shadow is 9 ft. Sebastian is 5 ft 10 in tall. Find the height of the box.
25. Juanita wonders how tall the mast of a ship she spots in the harbor is. The deck of the ship is the same height as the pier on which she is standing. The shadow of the mast is on the pier and she measures it to be 18 ft long. Juanita is 5 ft 4 in tall and her shadow is 4 ft long. How tall is the ship's mast?
26. Use shadows or a mirror to measure the height of an object in your yard or on the school grounds. Draw a picture to illustrate your method.

Use the diagram below to answer questions 27-31.



27. Draw the three separate triangles in the diagram.
28. Explain why $\triangle GDE \cong \triangle DFE \cong \triangle GFD$.

Complete the following proportionality statements.

29. $\frac{GF}{DF} = \frac{?}{FE}$
30. $\frac{GF}{GD} = \frac{?}{GE}$
31. $\frac{GE}{DE} = \frac{DE}{?}$

Review Queue Answers

a. $x = 52^\circ, y = 80^\circ$
b. $\frac{w}{20} = \frac{15}{25}$ $\qquad$ $\frac{15}{25} = \frac{18}{z}$

$25w = 15(20)$ $\qquad$ $25(18) = 15z$

$25w = 300$ $\qquad$ $450 = 15z$

$w = 12$ $\qquad$ $30 = z$

a. $168 = 8x$ $\qquad$ $6y = 216$

$x = 21$ $\qquad$ $y = 36$

b. Answers will vary. One possibility: $\frac{28}{8} = \frac{21}{6}$
c. $28(12) = 8(6+x)$

$336 = 48 + 8x$

$288 = 8x$

$36 = x$ Because $x \neq 21$, like in part a, this is not a true proportion.

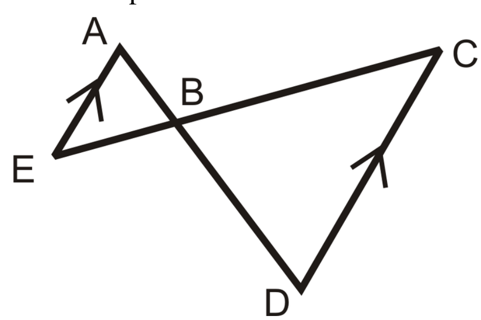
7.4 Similarity by SSS and SAS

Learning Objectives

- Use SSS and SAS to determine whether triangles are similar.
- Apply SSS and SAS to solve problems about similar triangles.

Review Queue

a. a. What are the congruent angles? List each pair.



b. Write the similarity statement.

c. If $AB = 8, BD = 20$, and $BC = 25$, find BE.

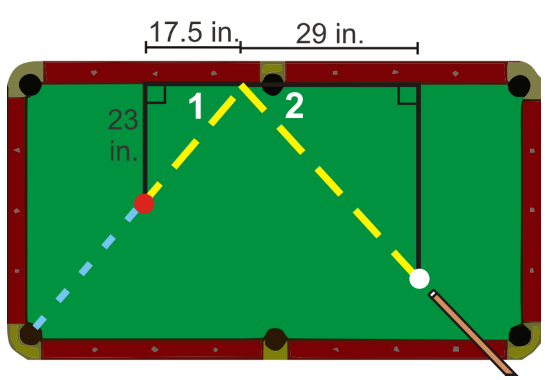
b. Solve the following proportions.

a. $\frac{6}{8} = \frac{21}{x}$

b. $\frac{x+2}{6} = \frac{2x-1}{15}$

c. $\frac{x-3}{9} = \frac{4}{x+2}$

Know What? Recall from Chapter 2, that the game of pool relies heavily on angles. In Section 2.5, we discovered that $m\angle 1 = m\angle 2$.



The dimensions of a pool table are 92 inches by 46 inches. You decide to hit the cue ball so it follows the yellow path to the right. The horizontal and vertical distances are in the picture. Are the two triangles similar? Why? How far did the cue ball travel to get to the red ball?

Link for an interactive game of pool: http://www.coolmath-games.com/0-poolgeometry/index.html

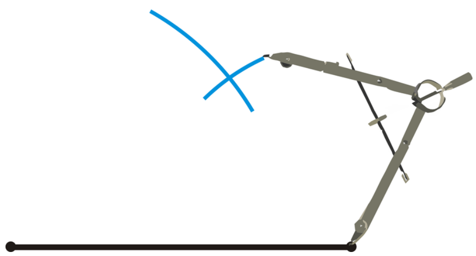
SSS for Similar Triangles

If you do not know any angle measures, can you say two triangles are similar? Let's investigate this to see. You will need to recall Investigation 4-2, Constructing a Triangle, given Three Sides.

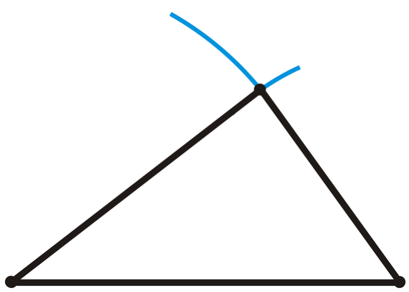
Investigation 7-2: SSS Similarity

Tools Needed: ruler, compass, protractor, paper, pencil

a. Using Investigation 4-2, construct a triangle with sides 6 cm, 8 cm, and 10 cm.



b. Construct a second triangle with sides 9 cm, 12 cm, and 15 cm.
c. Using your protractor, measure the angles in both triangles. What do you notice?
d. Line up the corresponding sides. Write down the ratios of these sides. What happens?

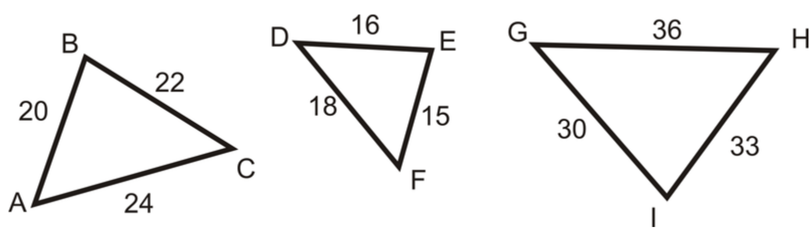


To see an animated construction of this, click: http://www.mathsisfun.com/geometry/construct-ruler-compass-1.htm l

From #3, you should notice that the angles in the two triangles are equal. Second, when the corresponding sides are lined up, the sides are all in the same proportion, $\frac{6}{9} = \frac{8}{12} = \frac{10}{15}$. If you were to repeat this activity, for a 3-4-5 or 12-16-20 triangle, you will notice that they are all similar. That is because, each of these triangles are multiples of 3-4-5. If we generalize what we found in this investigation, we have the SSS Similarity Theorem.

SSS Similarity Theorem: If the corresponding sides of two triangles are proportional, then the two triangles are similar.

Example 1: Determine if any of the triangles below are similar.



Solution: Compare two triangles at a time. In the proportions, place the shortest sides over each other, the longest sides over each other, and the middle sides over each other. Then, determine if the proportions are equal.

$\triangle ABC$ and $\triangle DEF$: $\frac{20}{15} = \frac{22}{16} = \frac{24}{18}$

Reduce each fraction to see if they are equal. $\frac{20}{15} = \frac{4}{3}, \frac{22}{16} = \frac{11}{8}$, and $\frac{24}{18} = \frac{4}{3}$. Because $\frac{4}{3} \neq \frac{11}{8}, \triangle ABC$ and $\triangle DEF$ are **not** similar.

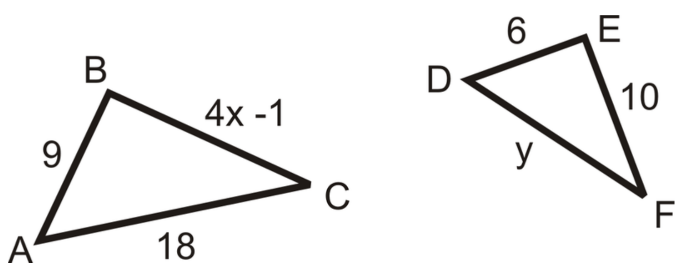
$\triangle DEF$ and $\triangle GHI$: $\frac{15}{30} = \frac{16}{33} = \frac{18}{36}$

$\frac{15}{30} = \frac{1}{2}, \frac{16}{33} = \frac{16}{33}$, and $\frac{18}{36} = \frac{1}{2}$. Because $\frac{1}{2} \neq \frac{16}{33}, \triangle DEF$ is not similar to $\triangle GHI$.

$\triangle ABC$ and $\triangle GHI$: $\frac{20}{30} = \frac{22}{33} = \frac{24}{36}$

$\frac{20}{30} = \frac{2}{3}, \frac{22}{33} = \frac{2}{3}$, and $\frac{24}{36} = \frac{2}{3}$. Because all three ratios reduce to $\frac{2}{3}, \triangle ABC \sim \triangle GIH$.

Example 2: ***Algebra Connection*** Find x and y, such that $\triangle ABC \sim \triangle DEF$.



Solution: According to the similarity statement, the corresponding sides are: $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$. Substituting in what we know, we have:

$$\frac{9}{6} = \frac{4x-1}{10} = \frac{18}{y}$$

$$\begin{aligned}\frac{9}{6} &= \frac{4x-1}{10} \\ 9(10) &= 6(4x-1) \\ 90 &= 24x-6 \\ 96 &= 24x \\ x &= 4\end{aligned}$$

$$\begin{aligned}\frac{9}{6} &= \frac{18}{y} \\ 9y &= 18(6) \\ 9y &= 108 \\ y &= 12\end{aligned}$$

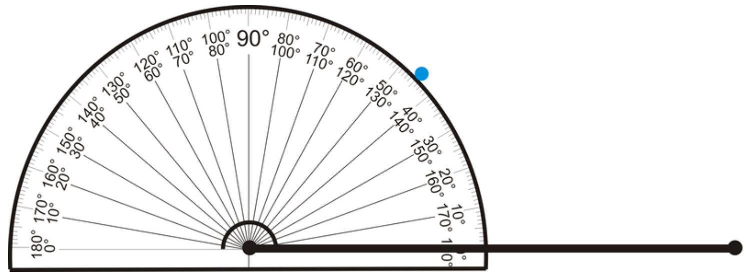
SAS for Similar Triangles

SAS is the last way to show two triangles are similar. If we know that two sides are proportional AND the included angles are congruent, then the two triangles are similar. For the following investigation, you will need to use Investigation 4-3, Constructing a Triangle with SAS.

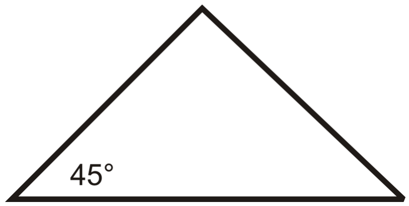
Investigation 7-3: SAS Similarity

Tools Needed: paper, pencil, ruler, protractor, compass

a. Using Investigation 4-3, construct a triangle with sides 6 cm and 4 cm and the *included* angle is 45°.



b. Repeat Step 1 and construct another triangle with sides 12 cm and 8 cm and the included angle is 45°.
c. Measure the other two angles in both triangles. What do you notice?

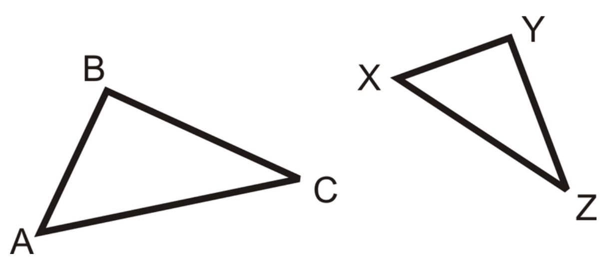


d. Measure the third side in each triangle. Make a ratio. Is this ratio the same as the ratios of the sides you were given?

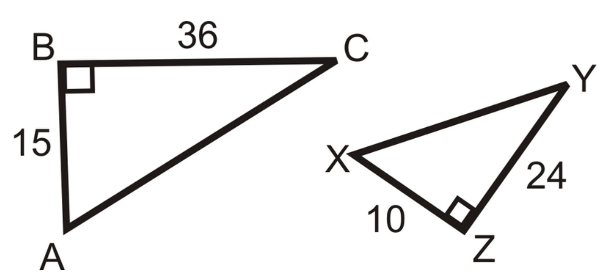
SAS Similarity Theorem: If two sides in one triangle are proportional to two sides in another triangle and the included angle in the first triangle is congruent to the included angle in the second, then the two triangles are similar.

In other words,

If $\frac{AB}{XY} = \frac{AC}{XZ}$ and $\angle A \cong \angle X$, then $\triangle ABC \sim \triangle XYZ$.

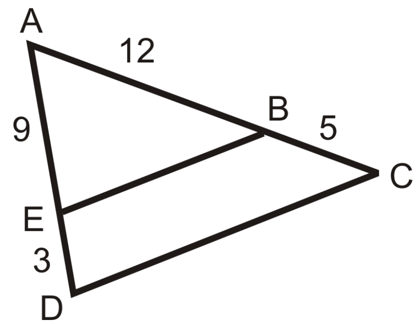


Example 3: Are the two triangles similar? How do you know?



Solution: $\angle B \cong \angle Z$ because they are both right angles. Second, $\frac{10}{15} = \frac{24}{36}$ because they both reduce to $\frac{2}{3}$. Therefore, $\frac{AB}{XZ} = \frac{BC}{ZY}$ and $\triangle ABC \sim \triangle XZY$.

Notice with this example that we can find the third sides of each triangle using the Pythagorean Theorem. If we were to find the third sides, $AC = 39$ and $XY = 26$. The ratio of these sides is $\frac{26}{39} = \frac{2}{3}$.



Example 4: Are there any similar triangles? How do you know?

Solution: $\angle A$ is shared by $\triangle EAB$ and $\triangle DAC$, so it is congruent to itself. If $\frac{AE}{AD} = \frac{AB}{AC}$ then, by SAS Similarity, the two triangles would be similar.

$$\frac{9}{9+3} = \frac{12}{12+5}$$
$$\frac{9}{12} = \frac{3}{4} \neq \frac{12}{17}$$

Because the proportion is not equal, the two triangles are not similar.

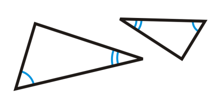
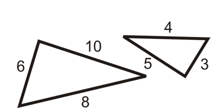
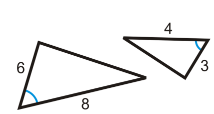
Example 5: From Example 4, what should BC equal for $\triangle EAB \sim \triangle DAC$?

Solution: The proportion we ended up with was $\frac{9}{12} = \frac{3}{4} \neq \frac{12}{17}$. AC needs to equal 16, so that $\frac{12}{16} = \frac{3}{4}$. Therefore, $AC = AB + BC$ and $16 = 12 + BC$. BC should equal 4 in order for $\triangle EAB \sim \triangle DAC$.

Similar Triangles Summary

Let's summarize what we've found that guarantees two triangles are similar.Two triangles are **similar** if and only if:

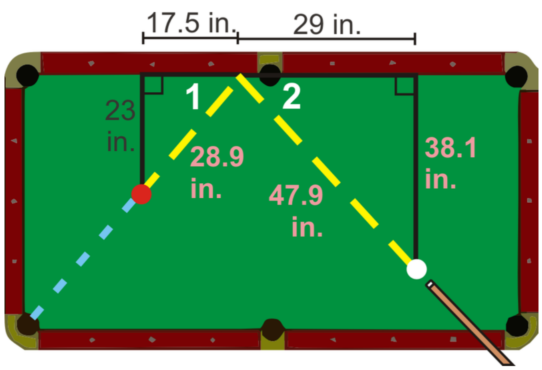
TABLE 7.2:

Name	*Description*	*Picture*
AA	Two angles in one triangle are congruent to two angles in another triangle.	
SSS for Similar Triangles	All three sides in one triangle are proportional to three sides in another triangle.	
SAS for Similar Triangles	Two sides in one triangle are proportional with two sides in another triangle AND the included angles are congruent.	

Know What? Revisited First, we need to find the vertical length of the larger triangle. The two triangles are similar by AA, two right angles and $\angle 1 \cong \angle 2$. Set up a proportion.

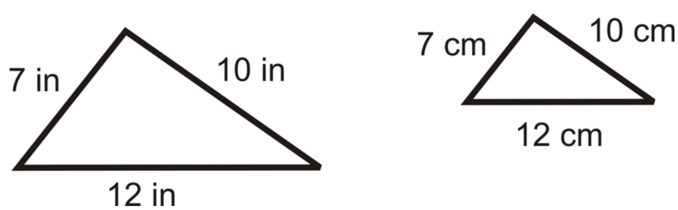
$$\frac{17.5}{23} = \frac{29}{v}$$

Doing cross-multiplication, $v = 38.1$. Second, to find the distance that the cue ball travels, use the Pythagorean Theorem. $17.5^2 + 23^2 = d_1^2$ and $38.1^2 + 29^2 = d_2^2$, the lengths 28.9 and 47.9, and the total length is 76.8 inches.



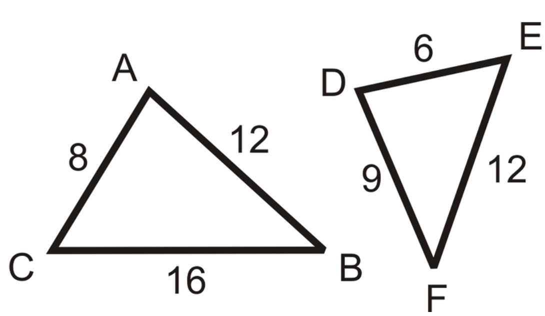
Review Questions

Use the following diagram for questions 1-3. *The diagram is to scale.*



1. Are the two triangles similar? Explain your answer.
2. Are the two triangles congruent? Explain your answer.
3. What is the scale factor for the two triangles?
4. ***Writing*** How come there is no ASA Similarity Theorem?

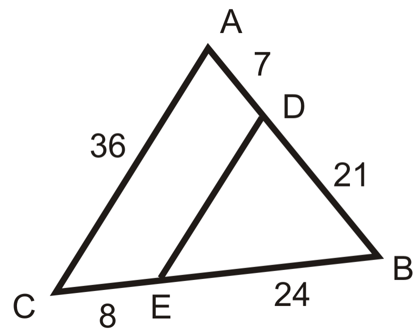
Fill in the blanks in the statements below. Use the diagram to the left.



5. $\triangle ABC \sim \triangle$____

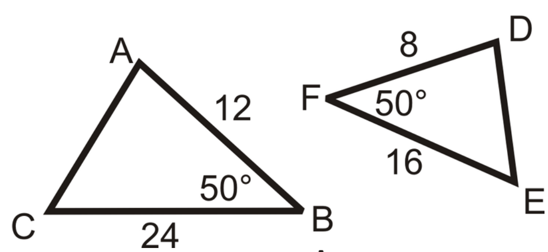
6. $\frac{AB}{?} = \frac{BC}{?} = \frac{AC}{?}$
7. If $\triangle ABC$ had an altitude, $AG = 10$, what would be the length of altitude $\overline{DH}$?

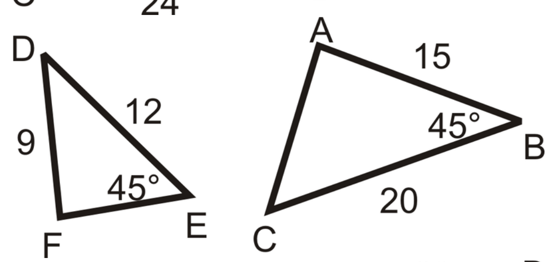
Use the diagram to the right for questions 8-12.

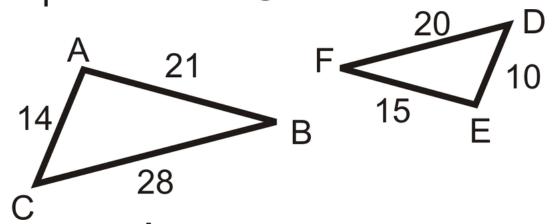


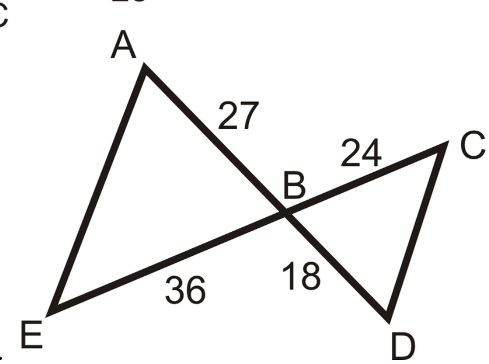
8. $\triangle ABC \sim \triangle$____
9. Why are the two triangles similar?
10. Find ED.
11. $\frac{BD}{?} = \frac{?}{BC} = \frac{DE}{?}$
12. Is $\frac{AD}{DB} = \frac{CE}{EB}$ a valid proportion? How do you know?

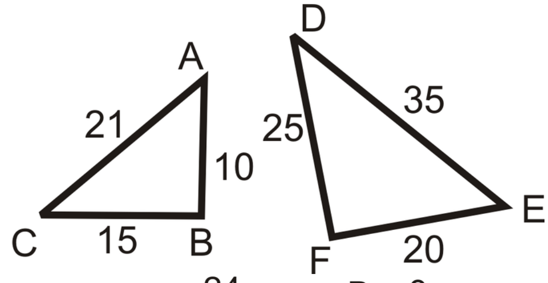
Determine if the following triangles are similar. If so, write the similarity theorem and statement.

13. 

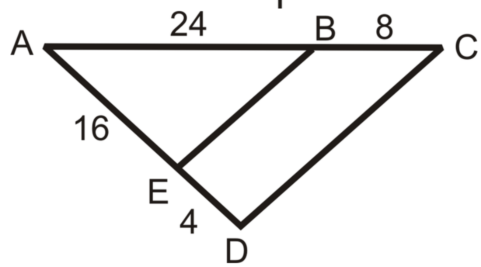
14. 

15. 

16. 

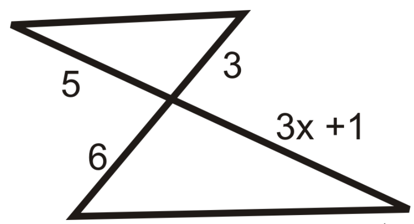


17.



18.

Algebra Connection Find the value of the missing variable(s) that makes the two triangles similar.



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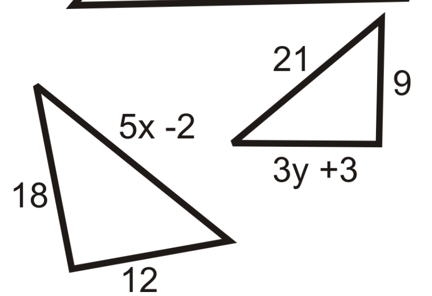


20.

21. At a certain time of day, a building casts a 25 ft shadow. At the same time of day, a 6 ft tall stop sign casts a 15 ft shadow. How tall is the building?
22. A child who is 42 inches tall is standing next to the stop sign in #21. How long is her shadow?
23. An architect wants to build 3 similar right triangles such that the ratio of the middle triangle to the small triangle is the same as the ratio of the largest triangle to the middle triangle. The smallest one has side lengths 5, 12 and 13. The largest triangle has side lengths 45, 108 and 117. What are the lengths of the sides of the middle triangle?
24. Jaime wants to find the height of a radio tower in his neighborhood. He places a mirror on the ground 30 ft from the tower and walks backwards 3 ft until he can see the top of the tower in the mirror. Jaime is 5 ft 6 in tall. How tall is the radio tower?

For questions 25-27, use $\triangle ABC$ with $A(-3,0), B(-1.5,3)$ and $C(0,0)$ and $\triangle DEF$ with $D(0,2), E(1,4)$ and $F(2,2)$.

25. Find AB, BC, AC, DE, EF and DF.
26. Use these values to find the following proportions: $\frac{AB}{DE}, \frac{BC}{EF}$ and $\frac{AC}{DF}$.
27. Are these triangles similar? Justify your answer.

For questions 28-31, use $\triangle CAR$ with $C(-3,3), A(-3,-1)$ and $R(0,-1)$ and $\triangle LOT$ with $L(5,-2), O(5,6)$ and $T(-1,6)$.

28. Find the slopes of $\overline{CA}, \overline{AR}, \overline{LO}$ and $\overline{OT}$.
29. What are the measures of $\angle A$ and $\angle O$? Explain.
30. Find LO, OT, CA and AR. Use these values to write the ratios $LO : CA$ and $OT : AR$.
31. Are the triangles similar? Justify your answer.

Review Queue Answer

a.
   a. $\angle A \cong \angle D, \angle E \cong \angle C$
   b. $\triangle ABE \sim \triangle DBC$
   c. $BE = 10$

   a. $\frac{6}{8} = \frac{21}{x}, x = 28$
   b. $$\begin{aligned} 15(x+2) &= 6(2x-1) \\ 15x+30 &= 12x-6 \\ 3x &= -36 \\ x &= -12 \end{aligned}$$

b. $$\begin{aligned} (x-3)(x+2) &= 36 \\ x^2-x-6 &= 36 \\ x^2-x-42 &= 0 \\ (x-7)(x+6) &= 0 \\ x &= 7, -6 \end{aligned}$$

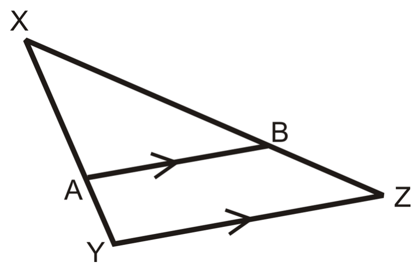
7.5 Proportionality Relationships

Learning Objectives

- Identify proportional segments when two sides of a triangle are cut by a segment parallel to the third side.
- Extend triangle proportionality to parallel lines.

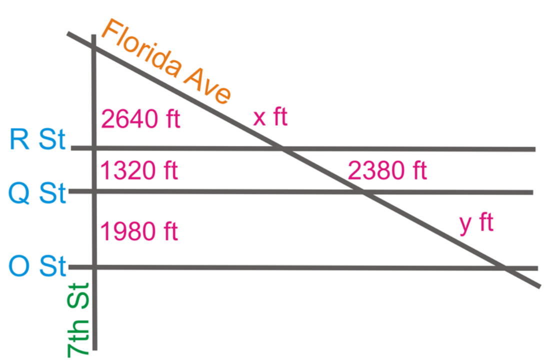
Review Queue

a. Write a similarity statement for the two triangles in the diagram. Why are they similar?



b. If $XA = 16, XY = 18, XB = 32$, find XZ.
c. If $YZ = 27$, find AB.
d. Find AY and BZ.
e. Is $\frac{AY}{AX} = \frac{BZ}{BX}$?

Know What? To the right is a street map of part of Washington DC. R Street, Q Street, and O Street are parallel and 7^{th} Street is perpendicular to all three. R and Q are one "city block" (usually $\frac{1}{4}$ mile or 1320 feet) apart. The other given measurements are on the map. What are x and y?



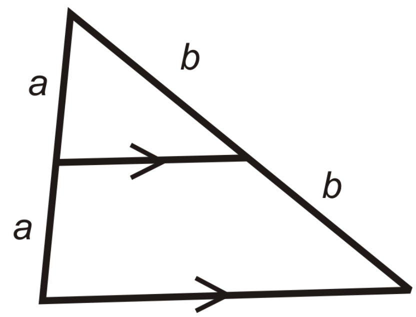
What is the distance from:

- R and 7^{th} to R and Florida?
- Q and 7^{th} to Q and Florida?
- O and 7^{th} to O and Florida?

Triangle Proportionality

Think about a midsegment of a triangle. A midsegment is parallel to one side of a triangle and divides the other two sides into congruent halves. The midsegment divides those two sides ***proportionally.***

Example 1: A triangle with its midsegment is drawn below. What is the ratio that the midsegment divides the sides into?



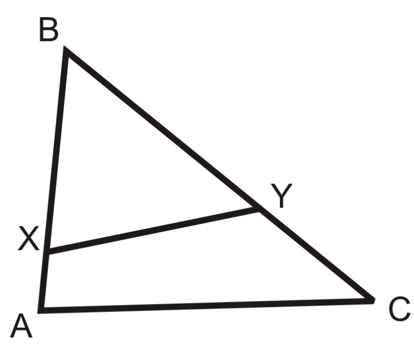
Solution: The midsegment's endpoints are the midpoints of the two sides it connects. The midpoints split the sides evenly. Therefore, the ratio would be $a:a$ or $b:b$. Both of these reduce to 1:1.

The midsegment divides the two sides of the triangle proportionally, but what about other segments?

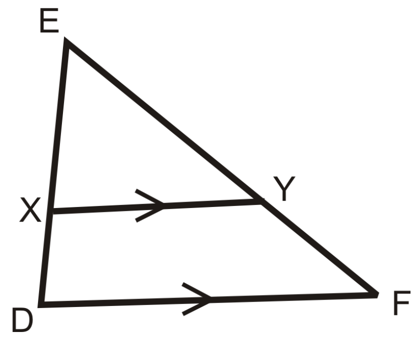
Investigation 7-4: Triangle Proportionality

Tools Needed: pencil, paper, ruler

a. Draw $\triangle ABC$. Label the vertices.
b. Draw $\overline{XY}$ so that X is on $\overline{AB}$ and Y is on $\overline{BC}$. X and Y can be ***anywhere*** on these sides.



c. Is $\triangle XBY \sim \triangle ABC$? Why or why not? Measure AX, XB, BY, and YC. Then set up the ratios $\frac{AX}{XB}$ and $\frac{YC}{YB}$. Are they equal?
d. Draw a second triangle, $\triangle DEF$. Label the vertices.
e. Draw $\overline{XY}$ so that X is on $\overline{DE}$ and Y is on $\overline{EF}$ AND $\overline{XY} \parallel \overline{DF}$.
f. Is $\triangle XEY \sim \triangle DEF$? Why or why not? Measure DX, XE, EY, and YF. Then set up the ratios $\frac{DX}{XE}$ and $\frac{FY}{YE}$. Are they equal?

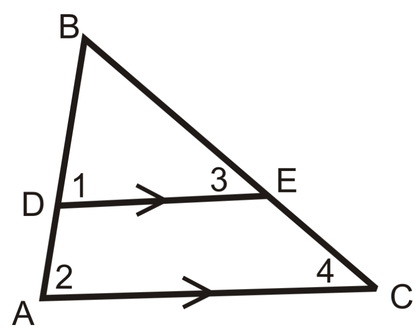


From this investigation, it is clear that if the line segments are parallel, then $\overline{XY}$ divides the sides proportionally.

Triangle Proportionality Theorem: If a line parallel to one side of a triangle intersects the other two sides, then it divides those sides proportionally.

Triangle Proportionality Theorem Converse: If a line divides two sides of a triangle proportionally, then it is parallel to the third side.

Proof of the Triangle Proportionality Theorem



Given: $\triangle ABC$ with $\overline{DE} \parallel \overline{AC}$

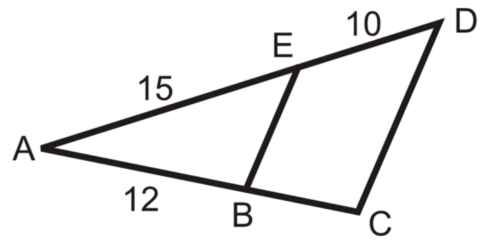
Prove: $\frac{AD}{DB} = \frac{CE}{EB}$

TABLE 7.3:

Statement	***Reason***
1. $\overline{DE} \parallel \overline{AC}$	Given
2. $\angle 1 \cong \angle 2, \angle 3 \cong \angle 4$	Corresponding Angles Postulate
3. $\triangle ABC \sim \triangle DBE$	AA Similarity Postulate
4. $AD + DB = AB$	
EC + EB = BC	Segment Addition Postulate
5. $\frac{AB}{BD} = \frac{BC}{BE}$	Corresponding sides in similar triangles are proportional
6. $\frac{AD+DB}{BD} = \frac{EC+EB}{BE}$	Substitution PoE
7. $\frac{AD}{BD} + \frac{DB}{DB} = \frac{EC}{BE} + \frac{BE}{BE}$	Separate the fractions
8. $\frac{AD}{BD} + 1 = \frac{EC}{BE} + 1$	Substitution PoE (something over itself always equals 1)
9. $\frac{AD}{BD} = \frac{EC}{BE}$	Subtraction PoE

We will not prove the converse, it is essentially this proof but in the reverse order. Using the corollaries from earlier in this chapter, $\frac{BD}{DA} = \frac{BE}{EC}$ is also a true proportion.

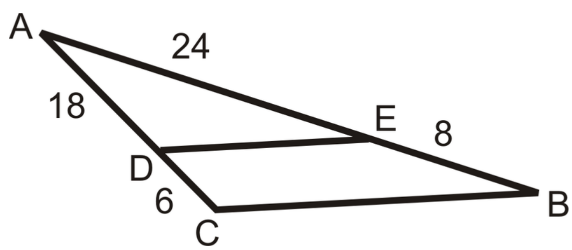
Example 2: In the diagram below, $\overline{EB} \parallel \overline{BD}$. Find BC.



Solution: Use the Triangle Proportionality Theorem.

$$\frac{10}{15} = \frac{BC}{12} \longrightarrow 15(BC) = 120$$
$$BC = 8$$

Example 3: Is $\overline{DE} \parallel \overline{CB}$?



Solution: Use the Triangle Proportionality Converse. If the ratios are equal, then the lines are parallel.

$\frac{6}{18} = \frac{1}{3}$ and $\frac{8}{24} = \frac{1}{3}$

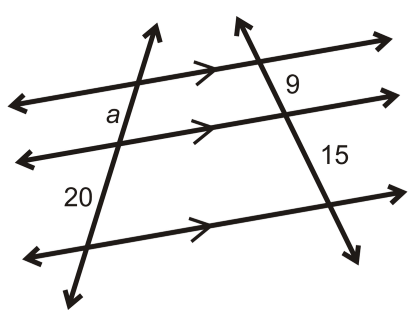
Because the ratios are equal, $\overline{DE} \parallel \overline{CB}$.

Parallel Lines and Transversals

We can extend the Triangle Proportionality Theorem to multiple parallel lines.

Theorem 7-7: If three parallel lines are cut by two transversals, then they divide the transversals proportionally.

Example 4: Find a.

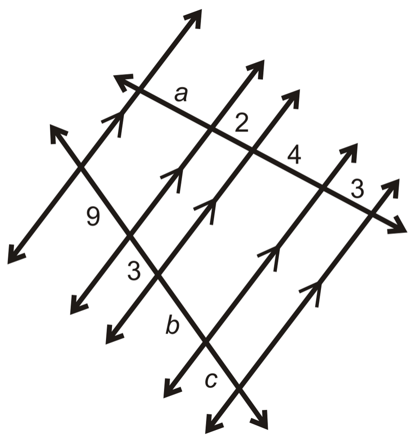


Solution: The three lines are marked parallel, so you can set up a proportion.

$$\frac{a}{20} = \frac{9}{15}$$
$$180 = 15a$$
$$a = 12$$

Theorem 7-7 can be expanded to ***any*** number of parallel lines with ***any*** number of transversals. When this happens all corresponding segments of the transversals are proportional.

Example 5: Find a, b, and c.

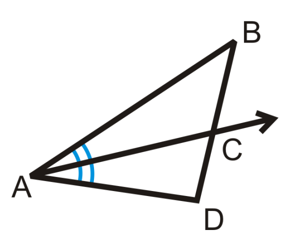


Solution: Look at the corresponding segments. Only the segment marked "2" is opposite a number, all the other segments are opposite variables. That means we will be using this ratio, 2:3 in all of our proportions.

$$\frac{a}{2} = \frac{9}{3} \qquad \frac{2}{4} = \frac{3}{b} \qquad \frac{2}{3} = \frac{3}{c}$$
$$3a = 18 \qquad 2b = 12 \qquad 2c = 9$$
$$a = 6 \qquad b = 6 \qquad c = 4.5$$

There are several ratios you can use to solve this example. To solve for b, you could have used the proportion $\frac{6}{4} = \frac{9}{b}$, which will still give you the same answer.

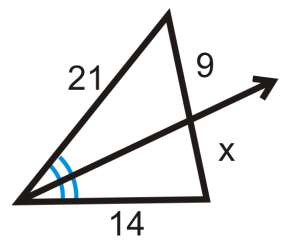
Proportions with Angle Bisectors



The last proportional relationship we will explore is how an angle bisector intersects the opposite side of a triangle. By definition, $\overrightarrow{AC}$ divides $\angle BAD$ equally, so $\angle BAC \cong \angle CAD$. The proportional relationship is $\frac{BC}{CD} = \frac{AB}{AD}$. The proof is in the review exercises.

Theorem 7-8: If a ray bisects an angle of a triangle, then it divides the opposite side into segments that are proportional to the lengths of the other two sides.

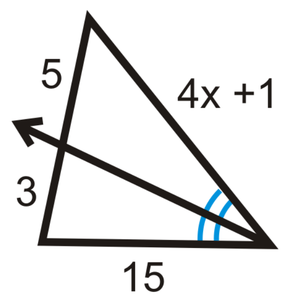
Example 6: Find x.



Solution: Because the ray is the angle bisector it splits the opposite side in the same ratio as the sides. So, the proportion is:

$$\frac{9}{x} = \frac{21}{14}$$
$$21x = 126$$
$$x = 6$$

Example 7: ***Algebra Connection*** Determine the value of x that would make the proportion true.



Solution: You can set up this proportion just like the previous example.

$$\frac{5}{3} = \frac{4x+1}{15}$$
$$75 = 3(4x+1)$$
$$75 = 12x+3$$
$$72 = 12x$$
$$6 = x$$

Know What? Revisited To find x and y, you need to set up a proportion using parallel the parallel lines.

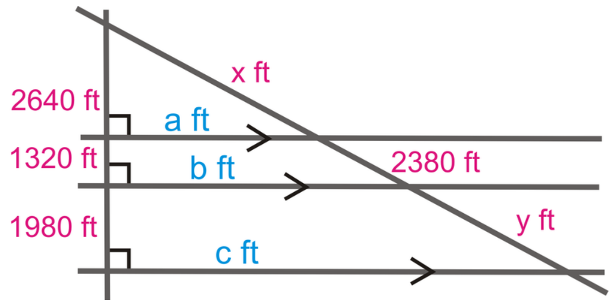
$$\frac{2640}{x} = \frac{1320}{2380} = \frac{1980}{y}$$

From this, $x = 4760\ ft$ and $y = 3570\ ft$.

To find a, b, and c, use the Pythagorean Theorem.

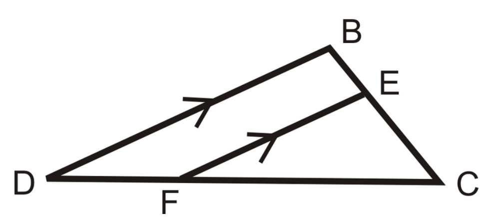
$$2640^2 + a^2 = 4760^2$$
$$3960^2 + b^2 = 7140^2$$
$$5940^2 + c^2 = 10710^2$$

$$a = 3960.81, \; b = 5941.21, \; c = 8911.82$$



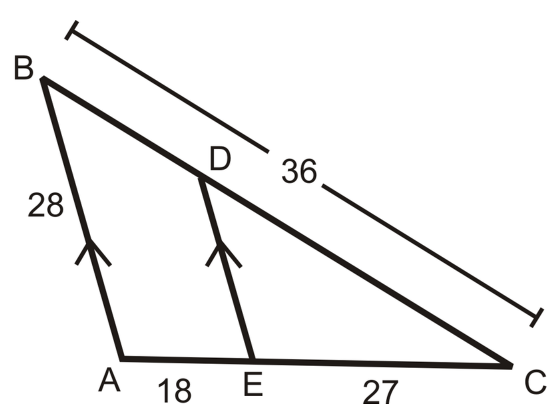
Review Questions

Use the diagram to answers questions 1-5. $\overline{DB} \,||\, \overline{FE}$.



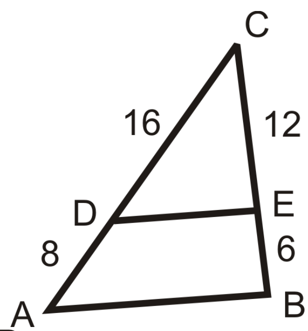
1. Name the similar triangles. Write the similarity statement.
2. $\frac{BE}{EC} = \frac{?}{FC}$
3. $\frac{EC}{CB} = \frac{CF}{?}$
4. $\frac{DB}{?} = \frac{BC}{EC}$
5. $\frac{FC+?}{FC} = \frac{?}{FE}$

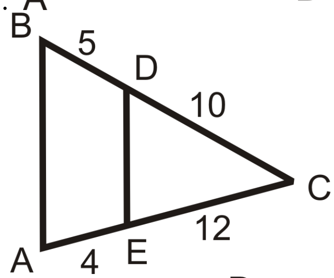
Use the diagram to answer questions 6-10. $\overline{AB} \,||\, \overline{DE}$.

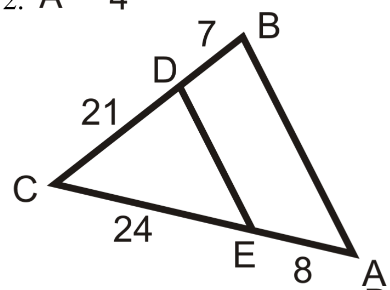


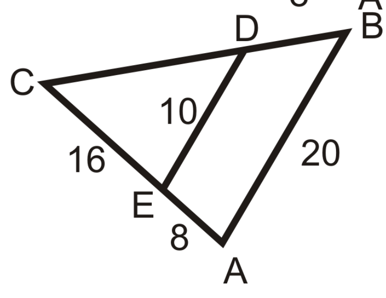
6. Find BD.
7. Find DC.
8. Find DE.
9. Find AC.
10. We know that $\frac{BD}{DC} = \frac{AE}{EC}$ and $\frac{BA}{DE} = \frac{BC}{DC}$. Why is $\frac{BA}{DE} \neq \frac{BD}{DC}$?

Use the given lengths to determine if $\overline{AB} \parallel \overline{DE}$.

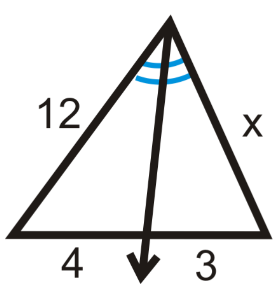
11. 

12. 

13. 

14. 

Algebra Connection Find the value of the missing variable(s).

15. 

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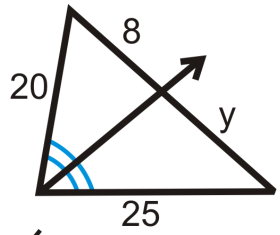

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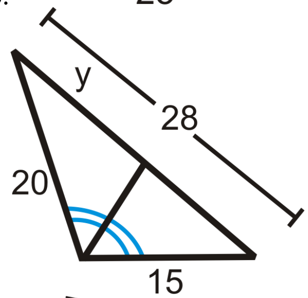

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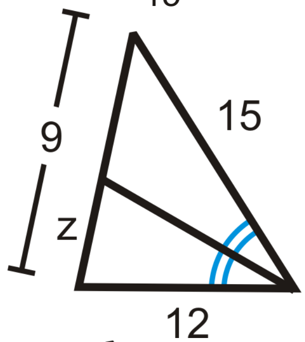

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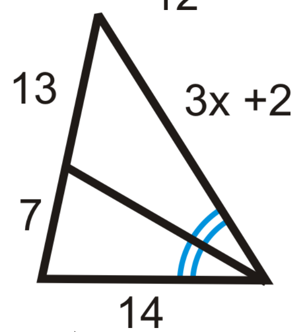

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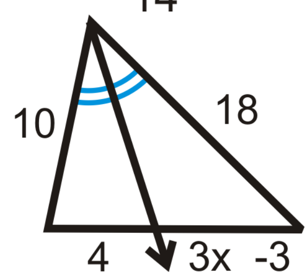
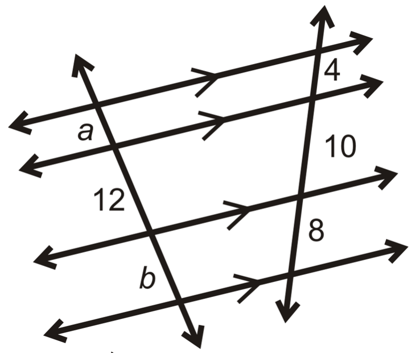
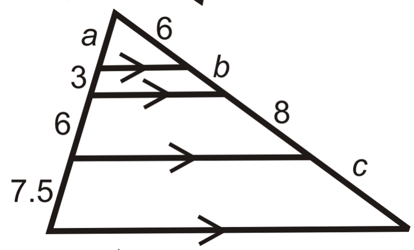
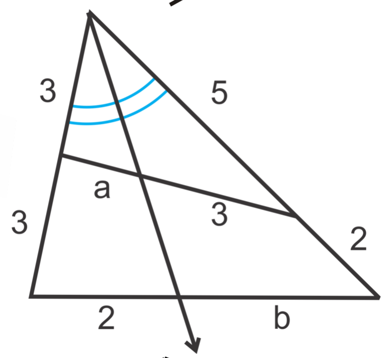
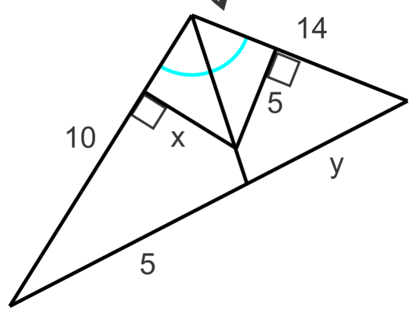


Find the value of each variable in the pictures below.

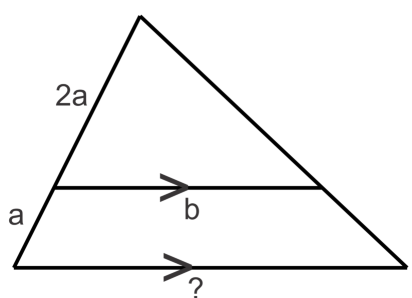
21. 

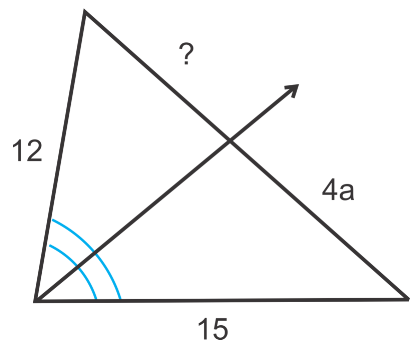
22. 

23. 

24. 

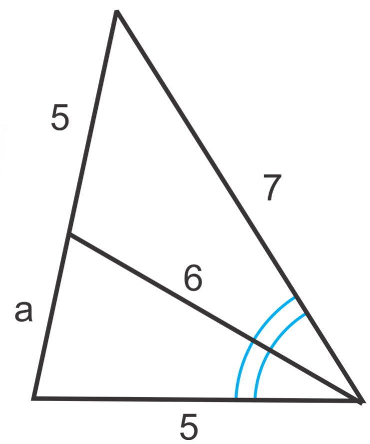
Find the unknown lengths.

25. 



26.

27. ***Error Analysis***

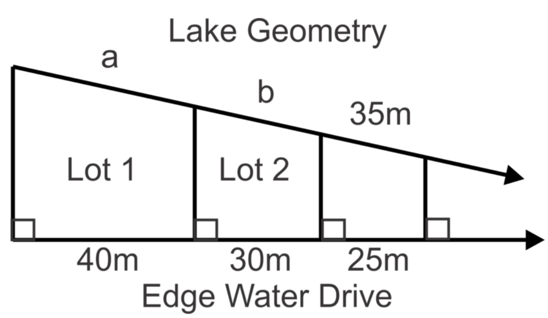


Casey attempts to solve for a in the diagram using the proportion

$$\frac{5}{a} = \frac{6}{5}$$

What did Casey do wrong? Write the correct proportion and solve for a.

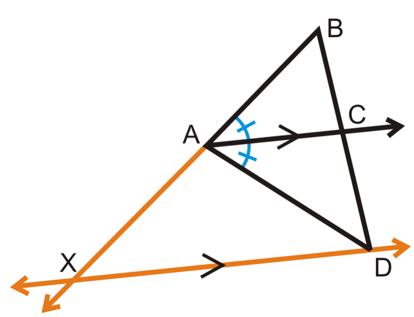
28. Michael has a triangular shaped garden with sides of length 3, 5 and 6 meters. He wishes to make a path along the perpendicular bisector of the angle between the sides of length 3 m and 5 m. Where will the path intersect the third side?



29.

This is a map of lake front properties. Find a and b, the length of the edge of Lot 1 and Lot 2 that is adjacent to the lake.

30. Fill in the blanks of the proof of Theorem 7-8.



Given: $\triangle BAD$ with $\overrightarrow{AC}$ is the angle bisector of $\angle BAD$ Auxiliary lines $\overrightarrow{AX}$ and $\overleftrightarrow{XD}$, such that X, A, B are collinear and $\overrightarrow{AC} \parallel \overleftrightarrow{XD}$. Prove: $\frac{BC}{CD} = \frac{BA}{AD}$

TABLE 7.4:

Statement	*Reason*
1. $\overrightarrow{AC}$ is the angle bisector of $\angle BADX, A, B$ are collinear and $\overrightarrow{AC} \parallel \overleftrightarrow{XD}$	
2. $\angle BAC \cong \angle CAD$	
3.	Corresponding Angles Postulate
4. $\angle CAD \cong \angle ADX$	
5. $\angle X \cong \angle ADX$	
6. $\triangle XAD$ is isosceles	
7.	Definition of an Isosceles Triangle
8.	Congruent segments are also equal
9.	Theorem 7-7
10.	

Review Queue Answers

a. $\triangle AXB \sim \triangle YXZ$ by AA Similarity Postulate
b. $\frac{16}{18} = \frac{32}{XZ}, XZ = 36$
c. $\frac{16}{18} = \frac{AB}{27}, AB = 24$
d. $AY = 18 - 16 = 2, BZ = 36 - 32 = 4$
e. $\frac{2}{16} = \frac{4}{32}$. Yes, this is a true proportion.

7.6 Extension: Self-Similarity

Learning Objectives

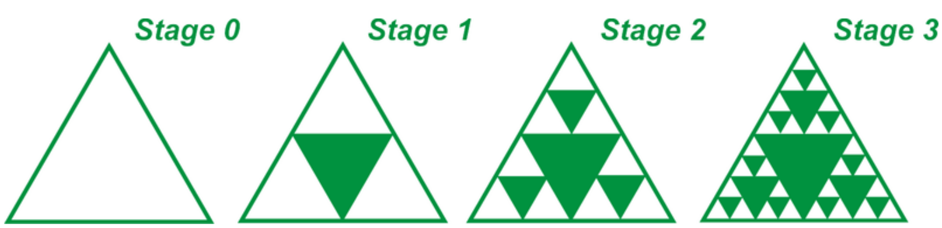
- Draw sets of the Sierpinski Triangle.
- Understand basic fractals.

Self-Similar: When one part of an object can be enlarged (or shrunk) to look like the whole object.

To explore self-similarity, we will go through a couple of examples. Typically, each step of repetition is called an iteration or level. The first level is called the Start Level or Stage 0.

Sierpinski Triangle

The Sierpinski triangle iterates an equilateral triangle (but, any triangle can be used) by connecting the midpoints of the sides and shading the central triangle (Stage 1). Repeat this process for the unshaded triangles in Stage 1 to get Stage 2. This series was part of the **Know What?** in Section 5.1.



Example 1: Determine the number of shaded and unshaded triangles in each stage of the Sierpinkski triangles. Determine if there is a pattern.

Solution:

TABLE 7.5:

	Stage 0	*Stage 1*	*Stage 2*	*Stage 3*
Unshaded	1	3	9	27
Shaded	0	1	4	13

The unshaded triangles seem to be powers of $3, 3^0, 3^1, 3^2, 3^3, \ldots$ The shaded triangles are add the previous number of unshaded triangles to the total. For Example, Stage 4 would equal $9 + 13$ shaded trangles.

Fractals

A fractal is another self-similar object that is repeated at successively smaller scales. Below are the first three stages of the Koch snowflake.

Example 2: Determine the number of edges and the perimeter of each snowflake.

TABLE 7.6:

	Stage 0	*Stage 1*	*Stage 2*
Number of Edges	3	12	48
Edge Length	1	$\frac{1}{3}$	$\frac{1}{9}$
Perimeter	3	4	$\frac{48}{9} = 5.\overline{3}$

The Cantor Set

The Cantor set is another fractal that consists of dividing a segment into thirds and then erasing the middle third.

Review Questions

1. Draw Stage 4 of the Cantor set.
2. Use the Cantor Set to fill in the table below.

TABLE 7.7:

	Number of Segments	*Length of each Segment*	*Total Length of the Segments*
Stage 0	1	1	1
Stage 1	2	$\frac{1}{3}$	$\frac{2}{3}$
Stage 2	4	$\frac{1}{9}$	$\frac{4}{9}$
Stage 3			
Stage 4			

TABLE 7.7: (continued)

	Number of Segments	*Length of each Segment*	*Total Length of the Segments*
Stage 5			

3. How many segments are in Stage n?
4. What is the length of each segment in Stage n?
5. Draw Stage 3 of the Koch snowflake.
6. Fill in the table from Example 2 for Stage 3 of the Koch snowflake.
7. A variation on the Sierpinski triangle is the Sierpinski carpet, which splits a square into 9 equal squares, coloring the middle one only. Then, split the uncolored squares to get the next stage. Draw the first 3 stages of this fractal.
8. How many shaded vs. unshaded squares are in each stage?
9. Fractals are very common in nature. For example, a fern leaf is a fractal. As the leaves get closer to the end, they get smaller and smaller. Find three other examples of fractals in nature.

10. Use the internet to explore fractals further. Write a paragraph about another example of a fractal in music, art or another field that interests you.

7.7 Chapter 7 Review

Keywords and Theorems

Ratio
A way to compare two numbers.

Proportion
When two ratios are set equal to each other.

Means
Mean (also called the arithmetic mean): The numerical balancing point of the data set. Calculated by adding all the data values and dividing the sum by the total number of data points.

Extremes
the product of the means must equal the product of the extremes.

Cross-Multiplication Theorem
the product of the means must equal the product of the extremes

Corollary
A theorem that follows quickly, easily, and directly from another theorem.

Corollary 7-1
If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

Corollary 7-2
Corollary 7-2 If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{d}{b} = \frac{c}{a}$.

Corollary 7-3
Corollary 7-3 If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

Corollary 7-4
Corollary 7-4 If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

Corollary 7-5
Corollary 7-5 If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$.

Similar Polygons
Two polygons with the same shape, but not the same size.

Scale Factor
In similar polygons, the ratio of one side of a polygon to the corresponding side of the other.

Theorem 7-2
The ratio of the perimeters of two similar polygons is the same as the ratio of the sides.

AA Similarity Postulate
If two angles in one triangle are congruent to two angles in another triangle, the two triangles are similar.

Indirect Measurement
An application of similar triangles is to measure lengths *indirectly.*

SSS Similarity Theorem
If the corresponding sides of two triangles are proportional, then the two triangles are similar.

SAS Similarity Theorem
If two sides in one triangle are proportional to two sides in another triangle and the included angle in the first triangle is congruent to the included angle in the second, then the two triangles are similar.

Triangle Proportionality Theorem
If a line parallel to one side of a triangle intersects the other two sides, then it divides those sides proportionally.

Triangle Proportionality Theorem Converse
If a line divides two sides of a triangle proportionally, then it is parallel to the third side.

Theorem 7-7
If three parallel lines are cut by two transversals, then they divide the transversals proportionally.

Theorem 7-8
If a ray bisects an angle of a triangle, then it divides the opposite side into segments that are proportional to the lengths of the other two sides.

Transformation
An operation that moves, flips, or changes a figure to create a new figure.

Rigid Transformation
Transformations that preserve size are ***rigid***

Non-rigid Transformation
Transformations that preserve size are ***rigid*** and ones that do not are ***non-rigid.***

Dilation
A non-rigid transformation that preserves shape but not size.

Self-Similar
When one part of an object can be enlarged (or shrunk) to look like the whole object.

Fractal
A fractal is another self-similar object that is repeated at successively smaller scales.

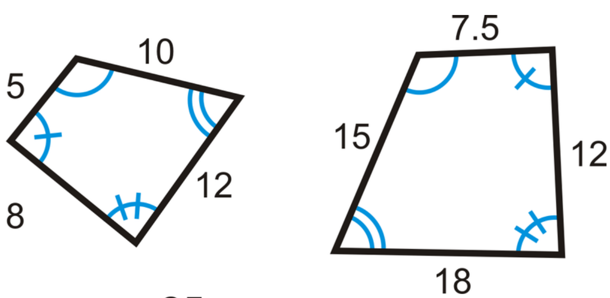
Review Questions

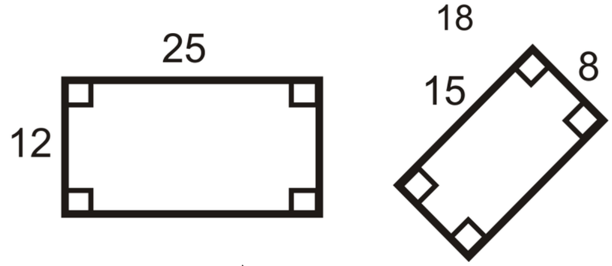
1. Solve the following proportions.

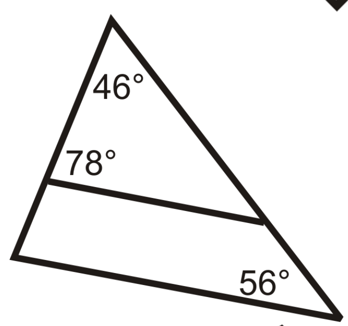
    a. $\frac{x+3}{3} = \frac{10}{2}$
    b. $\frac{8}{5} = \frac{2x-1}{x+3}$

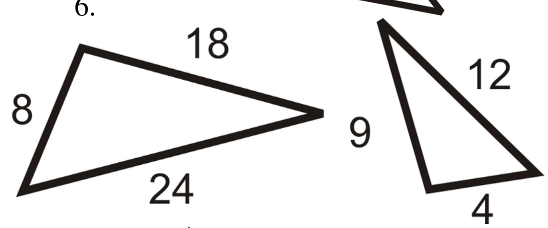
2. The extended ratio of the angle in a triangle are 5:6:7. What is the measure of each angle?
3. Rewrite 15 quarts in terms of gallons.

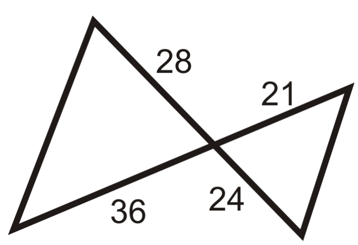
Determine if the following pairs of polygons are similar. If it is two triangles, write *why* they are similar.

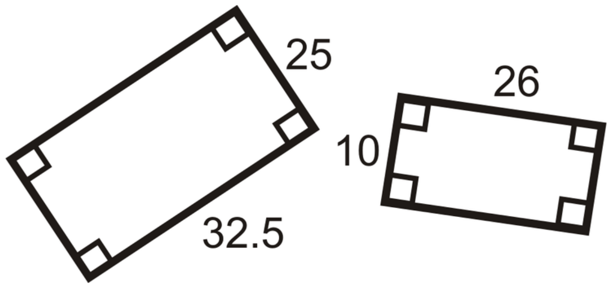
4. 

5. 

6. 

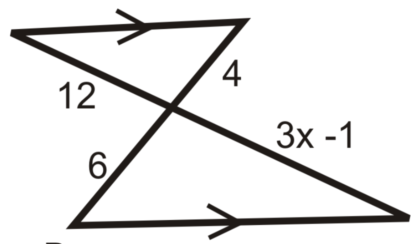
7. 

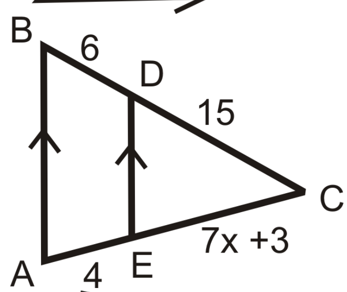
8. 

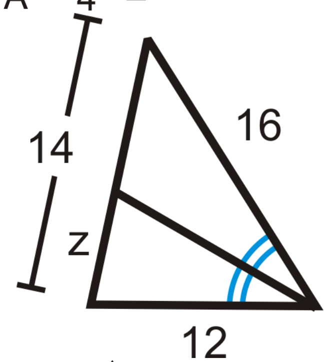
9. 

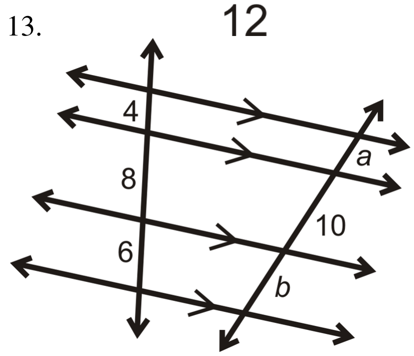
10. Draw a dilation of $A(7,2), B(4,9)$, and $C(-1,4)$ with $k = \frac{3}{2}$.

Algebra Connection Find the value of the missing variable(s).

11. 

12. 

13. 

14. 

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9692 .